1. The examination contains 3 problems. You have 75 minutes for 30 points.

2. Show all important steps in your work. Your answers will be graded on its correctness and clarity.

1. [10 points] Consider the following language:

\[ L = \{ x \in \{a, b\}^* \mid x \text{ has both } ab \text{ and } ba \text{ as substrings} \}. \]

(a) [5 points] Construct a deterministic finite automaton accepting the language \( L \). You may give its 5-tuple formal definition or its transition diagram. Give the idea of your construction, or brief and precise interpretations of the states of your machine.

(b) [5 points] Give a regular expression denoting the language \( L \). Annotate your regular expression or provide brief explanation for your answer.
2. [10 points] Consider the following two languages over $\Sigma = \{a, b\}$, 

$$
L_1 = \{x \in \Sigma^* | |\#_a(x) - \#_b(x)| \leq 2\}, \text{ and }
L_2 = \{x \in \Sigma^* | |\#_a(u) - \#_b(u)| \leq 2 \text{ for every prefix } u \text{ of } x\},
$$

where $\#_c(w)$ denotes the number of occurrences of the symbol $c$ in the string $w$.

Investigate the regularity of each of the languages. Prove your answers.

Notes:

- For proving regularity using finite automaton (deterministic or nondeterministic), the state-transition diagram and interpretation of the states of the machine are sufficient.
- For proving non-regularity, you must employ pumping lemma for regular languages.
3. [10 points]

(a) [5 points] Let $k$ be a positive integer. Prove that the language:

$$\{ x \in \{a,b\}^* \mid \#_a(x) \equiv 0 \pmod{k} \}$$

can not be accepted by any deterministic finite automaton with fewer than $k$ states.

Note that, for any integers $u$ and $v$, and positive integer $k$, we write $u \equiv v \pmod{k}$ to mean that the integer-division $\frac{u}{k}$ results in a remainder of $v$. 
(b) [5 points] It is known that the language of all “integer squares” over unary alphabet (i.e., \( \{a^i \mid i \geq 0\} \)) is not regular — proved in lecture and in textbook.

Let \( \Sigma \) denote the alphabet of decimal digits, i.e., \( \Sigma = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} \). Consider the language \( L \) of all “integer squares” over \( \Sigma \), that is,

\[
L = \{ x \mid x \text{ is an integer squared over } \Sigma \} \ (= \{0, 1, 4, 9, 16, 25, 36, 49, 64, 81, \ldots \}).
\]

Use closure properties and/or the pumping lemma for regular languages to prove that \( L \) is not regular. Hint: What is \( 101^2 \)?