1. The examination contains 3 problems. You have 75 minutes for 30 points.

2. Show all important steps in your work. Your answers will be graded on its correctness and clarity.

1. [10 points]
   
   (a) [2 points] Give an example language $L$ over an alphabet $\Sigma$ such that
   
   $$L \neq \emptyset, \text{ and } L^{2018} = L^{2019} \text{ and } L^0 \neq L^1.$$ 

   Explain your answer.

   (b) [4 points] Prove that for every non-empty language $L$ over an alphabet $\Sigma$,
   
   if $L^{2018} = L^{2019}$, then $L^0 \subseteq L^1$. 


(c) [4 points] A language is finite if it has finite cardinality/size. Prove that for every non-empty finite language $L$,

$$L^{2018} = L^{2019} \text{ if and only if } L^0 = L^1.$$ 

2. [15 points]

(a) [5 points] Construct the state-transition diagram of a deterministic finite automaton accepting the following language. Brief and precise interpretations of the states of your machine are required. Note: $\#_u(v)$ denotes the number of occurrences of a substring $u$ in a string $v$.

$$L = \{ x \in \{a, b\}^* \mid \#_a(x) + 2\#_b(x) \text{ is divisible by 3} \}.$$
(b) [10 points] For a given integer \( n \geq 2 \), let the alphabet \( \Sigma_n = \{a_1, a_2, \ldots, a_n\} \). Consider the following language:

\[
L_n = \{ x \in \Sigma_n^* \mid \text{there exists at least one symbol of } \Sigma_n \text{ not appearing in } x \}.
\]

For example, when \( n = 3 \) and \( \Sigma_3 = \{a, b, c\} \), then example strings \( \epsilon, a, bb, c, acaa \in L_3 \), but example strings \( bbac, aaaccebb \notin L_3 \).

i. [5 points] For \( n = 3 \), give the state-transition diagram of a finite automaton with at most 5 states that accepts the language \( L_3 \). A brief and precise interpretation of the states of your machine is required.
ii. [5 points] For every $n = 4$, show that every deterministic finite automaton accepting $L_4$ must have at least $2^4$ states.
3. [5 points] Let $\Sigma = \{0, 1\}$, $M = (Q, \Sigma, \delta, q_0, F)$ be a deterministic finite automaton, and $One(M) = \{x \mid x \in L(M), \text{ and } |x| \text{ is divisible by 3}\}$. Show that $One(M)$ is regular by giving explicitly the 5-tuple definition of a deterministic finite automaton accepting $One(M)$. 