1. The “closed-book/notes” examination contains 3 problems. You have 110 minutes for 30 points.

2. Show all important steps in your work. Your answers will be graded on its correctness and clarity.

1. [8 points]

(a) [2] Let \( k \) be an arbitrary positive integer. Give the state-transition diagram of a deterministic finite automaton \( M_k \) that accepts the language:
\[
L_k = \{ x \in \{a, b\}^* | \#_a(u) - \#_b(u) \leq k \text{ for every prefix } u \text{ of } x \},
\]
where \( \#_c(u) \) denotes the number of occurrences of the symbol \( c \) in the string \( u \). Give a brief and precise interpretation of the states in your construction.
Let $L$ be the language of all the strings of the form $(a^+b^+)^n$ such that $n$ is composite (that is, $n$ is not a prime integer). Formally,

$$L = \{x_1y_1x_2y_2\cdots x_ny_n \mid n \text{ is composite, and } x_i \in a^+ \text{ and } y_i \in b^+ \text{ for } i = 1, 2, \ldots, n\}.$$

Prove that: (1) $L$ is not regular, but (2) show that $L$ satisfies the pumping lemma for regular languages.
2. [12 points] Consider the language 

\[ L = \{(a^n b^n) \mid n \geq 1\}. \]

(a) [5] Show that \( L \) is not context-free.
(b) [7] Prove or disprove the context-freedom of the complement of $L$ (that is $\overline{L}$).
3. [10 points]

(a) [6] Consider the following language:

\[ E = \{ < M > | M \text{ is a deterministic finite automaton that accepts some string with more } 1\text{s than } 0\text{s} \}, \]

where \( < M > \) denotes an encoding of the deterministic finite automaton \( M \). Show that \( E \) is Turing-decidable.
(b) [4] Show that every infinite Turing-recognizable language $A$ has an infinite Turing-decidable language $B$ as subset. Note: The requirement of infinite cardinality of $B$ is important, as every finite language is regular — hence Turing-decidable.