2. Follow the procedure presented in Lecture 1, or the one in fact.

3. (a) Form no computational equivalence set of all regular expressions.

\[ \text{DFA} \equiv \text{FA} \equiv \text{RE} \]

\[ \text{NFA} = \text{NFA-ε} \]

It suffices to prove that:

\[ \forall \text{ regular expression } r \exists \text{ non-crossing FA } M \text{ such that } L(M) = L(r). \]

We can follow the proof of \( \text{RE} \subseteq \text{FA} \):

\[ \forall \text{ regular expression } r \exists \text{ FA } M \text{ such that } L(M) = L(r) \]

This was covered in Lecture (or Text) by using structural induction on \( \text{RE} \).

In order to prove (1) using structural induction on \( \text{RE} \), we need to strengthen the statement to:

\[ \forall \text{ regular expression } r \exists \text{ non-crossing FA } M \text{ with one start state } q_0 \text{ and one accept state } q_f \text{ satisfying } \]

\[ L(M) = L(r). \]

\[ \text{start} \rightarrow q_0 \rightarrow q_f \rightarrow \text{accept closure of } M \]

(b) Need to apply a known result in graph theory that gives a necessary condition of a graph \( G \) with \( n \) vertices and \( m \) edges: if \( G \) is non-crossing then

\[ m \leq 3n - 6. \]

Construct a regular language to violate the above necessary condition in (2).
4. Consider the language \( L = B - A \).
Since \( A \) and \( B \) are regular, so is \( L \).

Note that \( L \) is infinite. Applying Pumping Lemma to \( L \),
\( \exists z \in L \) such that we can write
\( z = uv^iw \) with \( |v| > 1 \),
and \( \forall i \geq 0 \), \( uv^iw \in L \).

Now, let \( C = A \cup \{ uv^iw \in L \mid i \text{ is odd} \} \).

Since the language \( \{ uv^iw \in L \mid i \text{ is odd} \} \) is regular,
the language \( C \) is also regular.

Since \( \{ uv^iw \in L \mid i \text{ is odd} \} - A \) is infinite, we have \( A \cup C \).

Similarly, \( C \cap B \).

5. (a) \( L_1 \) is regular:
Let \( \Sigma = \{0, 1\} \) and \( L' = \{ y \mid \exists x \in \Sigma^* \text{ such that } y \text{ contains at least one } 1 \} \).

Obviously \( L' \subseteq L_1 \) (\( n = 1 \)).

Claim: \( L' \) is regular and \( L_1 \subseteq L_1' \).

Since \( L_1' = \{ y \mid y \in \Sigma^* \text{ and } y \text{ contains at least two } 1 \} \),
we have \( L_1' = \{ 11 \} \cup \{ y \mid y \in \Sigma^* \text{ and } y \text{ contains at least one } 1 \} - \{ 1 \} \).

Since \( L_1 \) is regular,

To show \( L_1 \subseteq L_1' \), we consider arbitrarily...
(b) \(L_2\) is not regular:

Suppose \(L_2\) were regular. Let \(n \geq 1\) be the Pumping Lemma constant.

Consider \(z = 1^n 0^n \in L_2\) with \(|z| = 2(n+1) \geq n\).

Consider all \(u, v, w \in \{0, 1\}^*\) such that \(z = uvw\) and \(|w| \geq 1\).

Note that there is one major case

\[
\begin{array}{c}
\underline{u} \\
\underline{v} \\
\underline{w}
\end{array}
\]

so \(uv \in 1^*\) and \(v = 1^k\) for some \(1 \leq k \leq n\).

Now, for \(i = 0\), the unpumped string

\[uv^i w = uv^0 w = 1^{n-k} 0 1^n \notin L_2\]

Hence \(L_2\) is not regular.
6. Assume that the language \( L \) is regular.

A DFA \( M = (Q, \Sigma, \delta, q_0, F) \) accepts \( L \).

Without loss of generality, we may assume that \( \gamma \neq \varepsilon \).

Then, consider a string \( y^t \) of the form \( y^t \) for \( t = 1 \).

Now, feed all these strings to \( M \), and

find a sequence of states \( p_t \) with \( t \geq 1 \) such that \( \delta(q_0, y^t) = p_t \) for each \( t \geq 1 \).

Since \( M \) has finitely many states, by Pigeonhole Principle, eventually some state must be repeated. Therefore, \( \exists m, n \) with \( m > n \geq 1 \) such that \( \delta(q_0, y^m) = \delta(q_0, y^n) \).

By the deterministic nature of \( M \), we have, for every string \( x \), \( y^m x \) is accepted by \( M \) iff \( y^n x \) is also accepted by \( M \).

(a) \( L_1 = \{ 0^{i^2} | i > 0 \} \).

To show that \( L_1 \) is not regular, let \( y = 0 \) in the statement above.

Then, for all \( m, n \) with \( m > n \geq 0 \), let \( z = 0^{m^2 - n} \).

Clearly, \( y^m z = 0^{m^2} \in L_1 \).

For \( y^n z \), we have

\[ |y^n z| = n + m^2 - m < m + m^2 - m = m^2. \]

We also have \( |y^n z| = n + m^2 - m > m^2 - m = m(m-1) > (m-1)^2. \)

Therefore, \( (m-1)^2 < |y^n z| < m^2. \)
so \( y^n z \notin L_1 \). Here, \( L_1 \) is not regular.

(4) \[ L_2 = \{ uu^rv \mid u, v \in (ab)^+ \} \]

Given

Let \( y = 01 \) in the statement above.

For all integers \( m, n \) with \( m > n \geq 1 \), let \( z = (10)^n 0 \).

We have \( y^n z \in L_2 \) but \( y^m z \notin L_2 \).

Here, \( L_2 \) is not regular.