1. [6 points] Prove, or disprove by giving a counter-example with explanation, each of the following statements.

   (a) [2 points] Let $L$ be a language over some alphabet $\Sigma$ such that for every string $w \in L$, there exists a positive integer $k$ such that $|w| \leq k$. Then $L$ must be a regular language.

   (b) [2 points] Let $L_1$ and $L_2$ be languages that can be accepted by nondeterministic finite automata (with or without $\epsilon$-transitions) with $n_1$ and $n_2$ states, respectively. Then $L_1 \cap L_2$ is accepted by a deterministic finite automaton with at most $2^{n_1+n_2}$ states.

   (c) [2 points] It is possible that the intersection of an infinite number of regular languages is not regular.
2. [16 points] For each of the following languages $L$, decide if $L$ is regular and prove your answer. Note: If your argument involves construction of finite automaton or regular expression, you should provide brief/precise interpretation of the machine states or annotation, respectively.

(a) [4 points] $L = \{x\#y \mid x, y \in \{0, 1\}^*, \text{and when viewed as binary numbers}, x + y = 3y\}$. For examples, the strings 1000#100 and 1010#101 are in $L$ and 110#10 is not in $L$. 

(b) [4 points] \( L = \{ w \in \{a, b\}^* \mid \text{if } w \text{ contains the substring } ab, \text{ then } w \text{ contains the substring } ba \} \).
(c) [4 points] \( L = \{ xyzy \mid x, y, z \in \{0, 1\}^+ \} \).
(d) [4 points] $L = \{a^i b^j \mid i, j \geq 1, \text{ and either } (i \geq j) \text{ or } (i < j \text{ and } j \text{ is a multiple of } i)\}$. 
3. [8 points] Define a (language-)operator $\tau$ on languages as follows: for a language $L$ over an alphabet $\Sigma$,

$$\tau(L) = \{ w \in \Sigma^* \mid w \in L, \text{ and for every non-empty string } z \in \Sigma^*, wz \not\in L \}. $$

(a) [3 points] Assume the alphabet $\Sigma = \{a, b\}$ and the three languages: $L_1$ denoted by the regular expression $ba^*a$, $L_2 = \{ u \in \{a, b\}^* \mid u \text{ contains exactly one } a \}$, and $L_3 = \{a\}$. What are $\tau(L_1)$ and $\tau(L_2L_3)$ — with brief/precise explanations?

(b) [5 points] Prove that the operator $\tau$ preserves regularity — with detailed explanation.