Matching of 2-D Polygonal Arcs by using a Subgroup of the Unit Quaternions

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Abstract

A subgroup of the unit quaternions is used to calculate 2D rotations. Benefits of using quaternions over the more common methods include the ability to use the algebra of quaternions to find closed form solutions and the ability to use the same approach for both 2-D and 3-D algorithms. The subgroup is applied to matching polygonal arcs of equal length with the resulting solution being the smallest eigenvalue of a 2×2 matrix. This result is then used to match a short arc to locations on a long arc.

List of Symbols

θ

$\times$

$\sum$

1 INTRODUCTION

Matching of polygonal arcs is a basic problem in computer vision. The problem of finding an approximate match between short arcs and pieces of a long arc is known as the segment matching problem [4], [1]. This problem has found many applications, e.g., shape and object recognition, analysis of engineering drawings, character recognition, etc. ([1], [9], [11]). An algorithm has been developed in Parsi *et al.* [10] using a least-squares approach to match 2-D polygonal arcs where a distance measure is defined for arcs and the best match between two arcs is obtained by calculating, analytically, the relative position and orientation of the arcs that minimizes the distance measure. In this paper the calculations in that algorithm are simplified by finding a closed form solution to the minimum distance by using a subgroup of the unit quaternions. The minimum distance is the smallest eigenvalue of a 2×2 matrix. By using the full group of unit quaternions, matching of 3-D polygonal arcs may be done [6].

Planar rotations are normally represented by an angle of rotation around the origin. We have found it advantageous to use quaternions to represent planar rotations. As we are unaware of previous use of quaternions for planar rotations, we present an explanation of using quaternions for 2-D rotations in section 2. In section 3 we highlight the previous results from matching of 3-D arcs. Then in section 4 we derive new results for matching of 2-D arcs. In section 5 we show experimental results and section 6 is the conclusion.

2 Quaternions for 2-D rotations

First we will give a brief review of quaternions. We will then show how to use a subgroup of the unit quaternions to represent 2-D rotations.

Unit quaternions are often used to represent rotations on 3-D [3], [5]. A quaternion has four components, $\mathbf{q} = q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} + q_4$ which we shall represent as the vector $\mathbf{q} = [q_1, q_2, q_3, q_4]^T$. Addition of two quaternions is carried out by normal vector addition. Multiplication of two quaternions may be calculated by matrix multiplication with the following method [12]: given two quaternions $\mathbf{p}$ and $\mathbf{q}$ then the quaternion multiplication is $\mathbf{p} \times \mathbf{q} = \mathbf{R}(\mathbf{q})\mathbf{p}$ and is $\mathbf{q} \times \mathbf{p} = \mathbf{L}(\mathbf{q})\mathbf{p}$ where

$$\mathbf{L}(\mathbf{q}) = \begin{bmatrix} q_4 & -q_3 & q_2 & q_1 \\ q_3 & q_4 & -q_1 & q_2 \\ -q_2 & q_1 & q_4 & q_3 \\ -q_1 & -q_2 & -q_3 & q_4 \end{bmatrix} \qquad \mathbf{R}(\mathbf{q}) = \begin{bmatrix} q_4 & q_3 & -q_2 & q_1 \\ -q_3 & q_4 & q_1 & q_2 \\ q_2 & -q_1 & q_4 & q_3 \\ -q_1 & -q_2 & -q_3 & q_4 \end{bmatrix}$$

It is well known (e.g. [7, p. 438]) that rotating a point $\mathbf{v}$ in the 3-D space may be done by the quaternion multiplication $\mathbf{v} \rightarrow \mathbf{q}\mathbf{v}\overline{\mathbf{q}}$ where the conjugate $\overline{\mathbf{q}} = [-q_1, -q_2, -q_3, q_4]$. Thus the rotation matrix depending on $\mathbf{q}$ is just $\mathbf{Rot}(\mathbf{q}) = \mathbf{L}(\mathbf{q})\mathbf{R}(\overline{\mathbf{q}})$.

To use quaternions for 2-D rotations, we embed the 2-D plane in 3-D as the $z = 0$ plane. If we let Q_z be the set of unit quaternions of the form $[0, 0, a, b]$ for any real a and b, then Q_z is a subgroup of the unit quaternions, i.e., closed under quaternion multiplication ($\mathbf{q}_1 = [0, 0, a_1, b_1]$ and $\mathbf{q}_2 = [0, 0, a_2, b_2]$ then $\mathbf{q}_1\mathbf{q}_2 = [0, 0, a_1b_2 + a_2b_1, b_1b_2 - a_1a_2]$) and inverse ($\mathbf{q}_1^{-1} = \overline{\mathbf{q}_1} = [0, 0, -a_1, b_1]$). This subgroup represents the rotations around the z axis in the X-Y plane. To apply a rotation θ to a point $\mathbf{v} = (v_x, v_y)$ in the plane, first embed $\mathbf{v}$ as the quaternion $[v_x, v_y, 0, 0]$, next create the planar rotation quaternion $\mathbf{q} = [0, 0, \sin(\theta/2), \cos(\theta/2)]$, then $\mathbf{q}\mathbf{v}\overline{\mathbf{q}} = \mathbf{L}(\mathbf{q})\mathbf{R}(\overline{\mathbf{q}})\mathbf{v} = [v_x', v_y', 0, 0]$ where

$$\begin{aligned} v_x' &= v_x \cos(\theta/2)^2 - 2v_y \cos(\theta/2)\sin(\theta/2) - v_x \sin(\theta/2)^2 \\ v_y' &= v_y \cos(\theta/2)^2 + 2v_x \cos(\theta/2)\sin(\theta/2) - v_y \sin(\theta/2)^2 \end{aligned}$$

which is the rotation of $\mathbf{v}$ around the z axis by θ.

Thus we can use the algebra of quaternions to calculate 2-D rotations by using the subgroup Q_z.

3 Polygonal Arcs in 3-D

To make this paper self-contained, we shall review some results from our previous work. We will summarize the approach taken in [6].

A polygonal arc is defined by a set of points (the *vertices*), successive pairs of vertices are joined by line segments (the *sides*). We specify an orientation to each polygonal arc – so it has an "initial" point and a "final" point. Correspondence of points on two different arcs is defined as points which have the same arc length from the initial point of their respective arc. A distance measure between two arcs is defined as the integral of the Euclidean distance between corresponding points. The arcs are split into line segments so that corresponding segments have equal length. Then the distance measure $M(I,J)$ for arcs I and J can be shown to be

$$M(I,J) = \sum_{i=1}^{k} \left\{ u_i|\mathbf{a}_i - \mathbf{b}_i|^2 + \frac{u_i^3}{12}|\mathbf{c}_i - \mathbf{d}_i| \right\} \tag{1}$$

where there are k line segments on each arc with the i^{th} segment having length u_i, midpoint $\mathbf{a}_i$ and unit direction $\mathbf{c}_i$ for arc I, midpoint $\mathbf{b}_i$ and unit direction $\mathbf{d}_i$ for arc J.

A mismatch measure is found by keeping I fixed and determining the displacements $J \rightarrow J^{'}$ that give the minimum value for the distance measure $M(I,J^{'})$. For 3-D displacements, $M(I,J')$ is

$$M(I,J^{'}) = \sum_{i=1}^{k}\{u_i|\mathbf{a}_i - \mathbf{Rot}(\mathbf{q})\mathbf{b}_i + \mathbf{t}|^2 + \frac{u_i^3}{12}|\mathbf{c}_i - \mathbf{Rot}(\mathbf{q})\mathbf{d}_i|^2\} \tag{2}$$

where the displacement is determined by the rotation $\mathbf{Rot}(\mathbf{q})$ and the translation $\mathbf{t}$. To find the minimum distance, (2) can be rewritten in the form $M(I,J^{'}) = \mathbf{q}^T\mathbf{G}\mathbf{q}$ where $\mathbf{G}$ is a certain real symmetric 4×4 positive semidefinite matrix

$$\mathbf{G} = \sum_{i=1}^{k}\{u_i\mathbf{A}^T\mathbf{A} + \frac{u_i^3}{12}\mathbf{B}^T\mathbf{B}\} \tag{3}$$

with $\mathbf{A}$ and $\mathbf{B}$ defined as two 4×4 matrices

$$\mathbf{A} = \mathbf{L}(\mathbf{a}_i) - \mathbf{R}(\mathbf{b}_i) - \mathbf{L}(\mathbf{C}_I) + \mathbf{R}(\mathbf{C}_J)$$
$$\mathbf{B} = \mathbf{L}(\mathbf{c}_i) - \mathbf{R}(\mathbf{d}_i)$$

and $\mathbf{C_I}$ and $\mathbf{C_J}$ are the position vectors of the centroids of I and J respectively. The smallest eigenvalue of the matrix $\mathbf{G}$ given by (3) is the unique minimum value of the distance measure $M(I, J^{'})$ and the eigenvector $\mathbf{q}$ corresponding to the smallest eigenvalue gives the displacement $J \rightarrow J^{'}$ by the rotation matrix $\mathbf{Rot}(\mathbf{q})$ and the translation $\mathbf{t} = \mathbf{C_I} - \mathbf{Rot}(\mathbf{q})\mathbf{C_j}$.

Since either end of an arc can be the initial point, we identify one as the forward direction and the other as the reverse direction. The mismatch measure is the smallest eigenvalue of the matrix $\mathbf{G}$ when calculated for both the forward and reverse directions.

Segment Matching Algorithm

The mismatch measure compares arcs of equal length. To conduct the segment matching, an algorithm similar to the one given in Parsi *et al.* [10] was used. Given two polygonal arcs I and J where $|I| > |J|$, we wish to match J with all possible subarcs I^* of I where $|I^*| = |J|$. The essential idea of the matching algorithm is to slide the *short* arc J along the *long* arc I and for every position A_i along I, calculate the mismatch measure for the subarc I_i^* of I with the initial point A_i where $|I_i^*| = |J|$. After visiting all possible locations (once and only once) we can decide on the best match by taking the minimum of the mismatch measures at all the locations.

4 Polygonal Arcs in 2-D

By restricting $\mathbf{a}_i$, $\mathbf{b}_i$, $\mathbf{c}_i$, $\mathbf{d}_i$, $\mathbf{t}$ to the $z = 0$ plane and restricting $\mathbf{q}$ to the subgroup Q_z, (2) is a distance measure for 2-D arcs. To minimize the distance measure $M(I, J^{'})$ for 2-D arcs three theorems are developed. Theorem 1 states that the translation is determined by the rotation when we are interested in the extreme values of the distance measure. Theorem 2 simplifies the form of the distance measure equation. And Theorem 3 states that minimizing the distance is an eigenvalue problem with the minimum being the smallest eigenvalue of a 2×2 minor of $\mathbf{G}$. The proofs of Theorem 1 and Theorem 2 are straightforward from the results in [6].

Theorem 1 *For two 2-D polygonal arcs I and J of equal lengths in the same plane, given any planar rotation $\mathbf{Rot(q)}$, where $\mathbf{q}$ is of the form $[0, 0, q_3, q_4]^T$, there is a unique planar translation $\mathbf{t} = [t_x, t_y, 0]^T$ that, together with $\mathbf{Rot(q)}$, generates a displacement of J giving an extreme value of the distance measure M. The translation $\mathbf{t}$ is given by*

$$\mathbf{t} = \mathbf{C_I} - \mathbf{Rot(q)C_J} \tag{4}$$

where $\mathbf{C_I}$ and $\mathbf{C_J}$ are the position vectors of the centroids of I and J respectively.

Theorem 2 *For two 2-D polygonal arcs I, J of equal lengths in the same plane, the extreme values of the distance measure $M(I, J^{'})$ are given by*

$$M(I, J^{'}) = \mathbf{q}^T\mathbf{G}\mathbf{q} \tag{5}$$

where $\mathbf{G}$ is a certain real symmetric 4×4 positive semidefinite matrix, $\mathbf{q}$ is a unit quaternion of the form $[0, 0, q_3, q_4]^T$, $\mathbf{q}^T$ denotes the usual matrix transpose of $\mathbf{q}$ and $\mathbf{q}^T\mathbf{G}\mathbf{q}$ is evaluated by standard matrix multiplication.

After a change of coordinates so that the plane containing the arcs I and J is the $z = 0$ plane in $\mathbf{R}^3$, the matrix $\mathbf{G}$ is the same matrix used in 3-D matching and is given explicitly in (3). With this we can derive the following theorem to find the minimum distance measure and the corresponding displacement for 2-D polygonal arcs.

Theorem 3 *For two 2-D polygonal arcs I, J of equal lengths, the distance measure $M(I, J')$ has a unique minimum for all planar displacements $J \to J'$. Furthermore the smallest eigenvalue of the 2×2 minor of $\mathbf{G}$ located in the lower right quadrant of $\mathbf{G}$ is the unique minimum value of the distance measure $M(I, J')$ and its corresponding eigenvector $[e_1, e_2]$ when viewed as the quaternion $\mathbf{q} = [0, 0, e_1, e_2]^T$ gives the displacement $J \to J'$ by the rotation matrix $\mathbf{Rot}(\mathbf{q})$ and the translation (4).*

Proof: Since $\mathbf{q}^T\mathbf{q} = 1$, (5) may be written as $M(I, J') = \mathbf{q}^T\mathbf{G}\mathbf{q} + \lambda(1 - \mathbf{q}^T\mathbf{q})$. Taking the partial derivatives of M with respect to $\mathbf{q}$ and setting to zero for the extreme values, we get $\mathbf{G}\mathbf{q} = \lambda\mathbf{q}$. Since the polygonal arcs are planar, when the vertices and directions are embedded as imaginary quaternions, they are of the form $[x, y, 0, 0]$. Then it can be seen that $\mathbf{G}$ is of the form:

$$\mathbf{G} = \begin{bmatrix} G_{1,1} & G_{1,2} & 0 & 0 \\ G_{2,1} & G_{2,2} & 0 & 0 \\ 0 & 0 & G_{3,3} & G_{3,4} \\ 0 & 0 & G_{4,3} & G_{4,4} \end{bmatrix}$$

With the restriction of the quaternion $i\mathbf{q}$ to the form $[0, 0, q_3, q_4]^T$ then the eigenvalue equation $\mathbf{G}\mathbf{q} = \lambda\mathbf{q}$ can be reduced down to just the eigenvalues of the submatrix of $\mathbf{G}$ containing rows 3 to 4 and columns 3 to 4. Since this minor is a real symmetric positive semidefinite matrix, all of its eigenvalues will be nonnegative and the smallest eigenvalue minimizes the distance measure. $\square$

The minimum distance measure depends on the choice of the initial points of the two arcs I and J. Arbitrarily name one direction for the arc J as forward and the opposite direction for arc J as reverse. Keeping the choice of direction for I constant, calculate the minimum distance between I and both the forward and reverse directions of J (by Theorem 3, just the smallest eigenvalue of a 2×2 matrix). Let the mismatch measure between I and J be the minimum of these two values. With this mismatch measure, the algorithm described in section 3 can be used for segment matching of 2-D arcs.

5 IMPLEMENTATION

We have tested the matching algorithm on a number of synthetic images. First we demonstrate the mismatch measure for two equal length 2-D polygonal arcs. Then we conduct the matching of 2-D arcs.

Given two equal length 2-D polygonal arcs I and J in Figure 1(a), the mismatch measure may be calculated as follows. Since either endpoint of J may be used as the initial vertex of the arc, arbitrarily name one endpoint as forward and the other as reverse. Then, for each direction, the matrix $\mathbf{G}$ (from Theorem 3) may be reduced to

$$\mathbf{G}_{\text{for}} = \begin{bmatrix} 50.67 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 42.67 & 10.67 \\ 0 & 0 & 10.67 & 8 \end{bmatrix} \qquad \mathbf{G}_{\text{rev}} = \begin{bmatrix} 5.33 & 0 & 0 & 0 \\ 0 & 45.33 & 0 & 0 \\ 0 & 0 & 8 & 16 \\ 0 & 0 & 16 & 42.67 \end{bmatrix}$$

The smallest eigenvalue of the lower right quadrant of $\mathbf{G}_{\text{for}}$ and $\mathbf{G}_{\text{rev}}$ is approximately 4.98 and 1.74 respectively. Using the eigenvector corresponding to the smaller eigenvalue, the quaternion to rotate J is $[0, 0, 0.931, -0.364]^T$ and the translation is $[4.51, 8.92]^T$. The resulting J' is superimposed over I in Figure 1(b). Note, if the matching was done using the 3-D mismatch measure then the smallest eigenvalue of the full matrices $\mathbf{G}_{\text{for}}$ and $\mathbf{G}_{\text{rev}}$ would have been used. In that case the eigenvalue is 0 which means an exact match (rotate J around the y axis and translate to I). Thus the arcs I and J do not match under 2-D but do match under 3-D.

[Figure 1 about here.]

To match 2-D arcs, I and J, we embed the plane containing the arcs in $\mathbf{R}^3$ as the $z = 0$

plane. Then the algorithm is applied using the mismatch measure from Theorem 3. For an example, let I be the long arc and J be the short arc in Figure 2(a). The minimum distance measure is calculated as we slide the short arc J along the longer arc I. The resulting graph of the minimum distance measure versus starting position on I is presented in Figure 2(b). Transforming J to J' for the lengths A,B,C,D, and E from Figure 2(b) and superimposing over the original arc I yields Figure 2(c). This figure verifies that the minimum value at length D in the histogram corresponds to the best match in Figure 2(c).

[Figure 2 about here.]

6 CONCLUSIONS

We have shown that the subgroup of the unit quaternions which represent rotations around the z-axis can be used to represent 2-D rotations. The benefits of using this subgroup include the ease of finding close formed solutions and the ability to use the same approach for 2-D and 3-D algorithms. We have used this subgroup in the segment matching problem of 2-D arcs. It allowed us to use basically the same theorems from the 3-D approach in 2-D matching. The resulting solution of the mismatch measure is simply the smallest eigenvalue of a 2×2 matrix. Since a curve may be approximated by a polygonal arc up to any degree of accuracy, these results may be used to match 2-D curves.

7 ACKNOWLEDGMENTS

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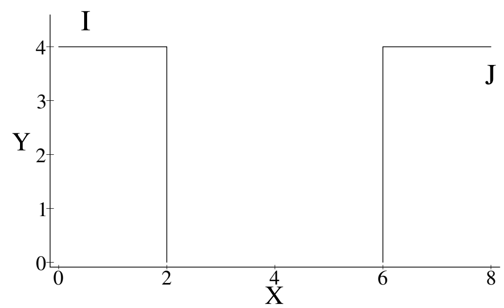
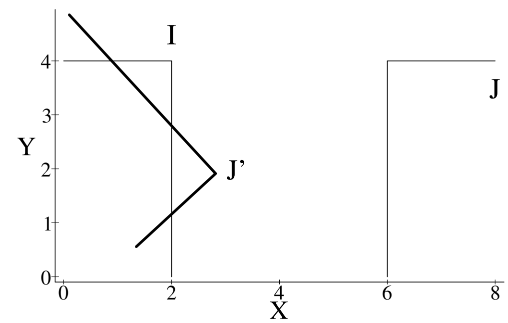
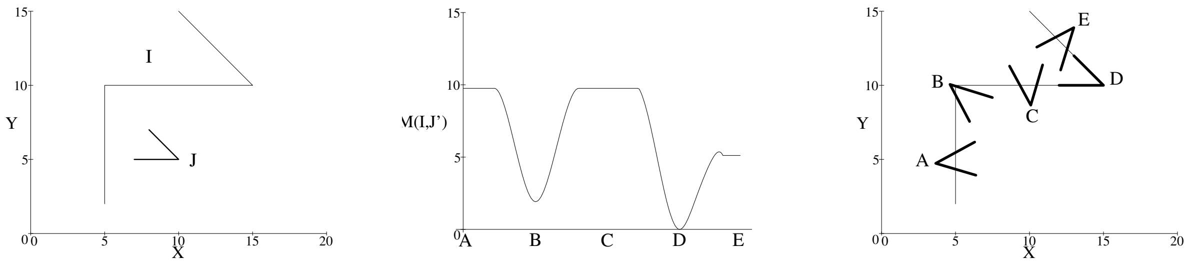


(a) Equal Length I and J



(b) Minimum mismatch measure arc J'.

Figure 1: Mismatch Measure for Equal Length Arcs



(a) Original Arcs I and J

(b) Distance measure $M(I, J^{'})$ versus Starting length on I

(c) I and a variety of $J^{'}$'s

Figure 2: Matching 2-D Polygonal Arcs