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Invariants of Families of Coplanar Conics and their Applications to Object Recognition

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Received ??. Revised ??.

Abstract.
This paper presents mutual invariants of families of coplanar conics. These invariants are compared with the use of invariants of two conics and a case is presented where the proposed invariants have a greater discriminating power than the previously used invariants. The use of invariants for two conics is extended to any number of coplanar conics. A lambda-matrix is associated with each family of coplanar conics. The use of lambda-matrices is extended from the single variable polynomial to multi-variable polynomials. The Segre characteristic and other invariants of the lambda-matrix are used as invariants of the family of conics.

Keywords: Invariants, Conics, Lambda Matrices, Segre Characteristics, Object Recognition

1. INTRODUCTION

Conics are widely recognized in the study of machine vision as the most fundamental image features next to lines. Many natural and man-made objects have circular shapes, and in addition, many other curves can be approximated by conics. Arbitrary planar shapes can be represented by a set of coplanar conics [3], [14]. There has been much work done in recognizing pairs of coplanar conics using invariants [8], [11], [17], [19], [24] and some current work on the invariants of three coplanar conics [33]. This paper extends the use of invariants to families of coplanar conics of any size. The proposed invariants can discriminate between two non-projectively related families of coplanar conics in which the previously used invariants give the same value.

Invariants are properties of geometric configurations which remain unchanged under an appropriate class of transformations. In fact, it has been asserted that invariance is the essential property of a shape description [7]. Mutual invariants are properties of a set of objects that remain unchanged under a class of transformations. Cooper *et. al.* [4] used mutual invariants in recognition of objects in aerial photographs. In a similar vein, the mutual invariants of a family of coplanar conics can be used in recognizing objects by being shape descriptors to index a model of the object from a database. Previous work with invariants from conics have focused exclusively on two coplanar conics. Often, an object will have more than two coplanar conics. This then leads to the question of using mutual invariants of more than two coplanar conics. In this paper we address that question by extending the use of invariants to recognizing objects with any number of coplanar conics.

The invariants we are interested in are the *projective invariants.* A projective invariant is an attribute of an object that will remain the same under changes in pose and camera calibrations since perspective transformations are contained in the set of projective transformations. For background material on invariant theory see [28], [10], [5], and [31]. For background on projective geometry see [30] and [32]. For background on the application

* This work was partially supported by the US Air Force Office of Scientific Research under Grant AFOSR 9620-92-J-286 and Grant AFOSR F49620-94-1-0029

of invariants to computer vision see [27], [8], [19], [20], and [25].

The plan of the paper is as follows. In section 2 we present some results on lambda-matrices. First in subsection 2.1, we review some of the known results for lambda-matrices in one variable. Then in subsection 2.2, we extend the results to lambda-matrices in several variables. In section 3 we associate a family of coplanar conics with a lambda-matrix and propose using the invariants of the lambda-matrix as invariants of the family of conics. In section 4 we compare the use of these invariants with previously used invariants and we present an example where the previous invariants do not discriminate between two non-projectively related families of conics and ours does. In section 5 the experimental results are presented. A method to compare values of the invariants using a probability distribution function is used to investigate the affects of noise in subsection 5.1. The invariants of three and four coplanar conics are used to match objects in subsection 5.2. The invariants are used to recognize tracked vehicles in subsection 5.3. In section 6 we summarize our paper and present directions of future research. The proofs of section 2 and 3 are located in the Appendix.

2. $\boldsymbol{\lambda}$-MATRICES

A matrix with polynomial entries is called a *lambda-matrix* ([2], [13], [15]). The lambda-matrix has been used in physics [9]. We will use a lambda-matrix as a tool for calculating the invariants of a family of coplanar conics. In this paper, the polynomials will be in the indeterminates $\lambda_1, \lambda_2, \ldots, \lambda_n$ where n will depend on the problem at hand, and the coefficients of the polynomials will lie in the field of complex numbers. When the polynomial entries are of a single variable λ we will refer to a lambda-matrix as a λ-*matrix* . When the polynomial entries are multi-variable, $\boldsymbol{\lambda} = [\lambda_1, \lambda_2, \ldots, \lambda_n]$, then we will refer to a lambda-matrix as a $\boldsymbol{\lambda}$-*matrix.* When needed, we shall make a restriction to using only 3×3 symmetric $\boldsymbol{\lambda}$-matrices with polynomials of degree at most 1. This is done because we will construct 3×3 symmetric lambda-matrices of first degree where the number of indeterminates is the number of conics in the family of coplanar conics.

First we review some of the known results for single variable λ-matrices over a field. Then we extend the results to multi-variable 3×3 symmetric $\boldsymbol{\lambda}$-matrices.

2.1. ***Uni-variable*** *λ-MATRICES*

If the entries of the λ-matrices are restricted to being single variable polynomials with coefficients lying in a field K, then a number of standard results are available [2], Chap. XX, [13], Chapter II, section 9, [15]. We shall use λ as the indeterminate in the single variable polynomials. A $n \times n$ λ-matrix $\mathbf{A}$ of rank k can be diagonalized to a canonical form:

$$\begin{bmatrix} E_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & E_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & E_k & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & & & & & & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where each E_i $(1 \leq i \leq k)$ is either equal to 1 or a monic polynomials in the indeterminate λ, and E_i divides E_{i+1} for $1 \leq i \leq k-1$. The E's are called the *invariant factors* of $\mathbf{A}$. If we take the field K to be the complex numbers, then factoring the E's into powers of linear polynomials will give the *elementary divisors* of $\mathbf{A}$.

The determinant of $\mathbf{A}$ is a constant multiple of the product of its invariant factors. If we express the determinant of $\mathbf{A}$ into a monic polynomial, then it is equal to the product of the invariant factors. We shall denote the elementary divisors of $\mathbf{A}$ by

$$(\lambda + \alpha_1)^{e_1}, (\lambda + \alpha_2)^{e_2}, \cdots, (\lambda + \alpha_k)^{e_m}$$

So, the linear factors $(\lambda + \alpha_i)$ need not be distinct from one another. The degrees $e_1, \ldots, e_m$, may be written as a symbol $[e_1\ e_2\ \ldots\ e_m]$. The degrees may be organized by grouping those which correspond to the same linear factor within parentheses and sorting in a non-ascending order. For example, if the elementary divisors of a matrix $\mathbf{A}$

are:

$$(\lambda - 5), (\lambda + 3), (\lambda + 3)^2$$

then the symbol would be [(2 1) (1)], and dropping the parentheses around single numbers, [(2 1) 1]. This symbol is called the *Segre characteristic* of $\mathbf{A}$ (see [29], page 227 or [32], page 188).

A λ-matrix is called *regular* if its determinant is a constant. Two λ-matrices $\mathbf{A}$ and $\mathbf{B}$ are *equivalent*, $\mathbf{A} \sim \mathbf{B}$, if there exist two regular λ-matrices $\mathbf{P}$ and $\mathbf{Q}$ such that $\mathbf{B} = \mathbf{PAQ}$. It is a standard result that two λ-matrices are equivalent if and only if they have the same rank and the same invariant factors. Two λ-matrices are equivalent if and only if they have the same rank and same elementary divisors. Two equivalent λ-matrices will have the same Segre characteristic, but two λ-matrices with the same Segre characteristic need not be equivalent.

We may add a straightforward result when the matrix is restricted to the size 3×3. The proof of this results can be found in the Appendix. This result will be used in the analysis of families of two coplanar conics in section IV.

PROPOSITION 1 *Two* 3×3 *λ-matrices are equivalent if and only if the have the same monic determinant and the same Segre characteristic.*

2.2. Multi-variable $\boldsymbol{\lambda}$-MATRICES

We shall now extend some of the results of subsection II.A from the single variable case to a multi-variable case. This extension does not appear in the literature and is, we believe, new. The polynomials will be in the indeterminates $\boldsymbol{\lambda} = [\lambda_1, \lambda_2, \ldots, \lambda_n]$. In moving to using multi-variable polynomials from the single variable polynomials, we are moving from an *Euclidean Domain* to an *Unique Factorization Domain* [6], chapter 8. Now we can no longer use the standard algorithm [13], pp. 89-91 for diagonalizing the single variable λ-matrix. Instead we define a diagonal matrix as a *standard form* which is similar to the canonical form of single variable λ-matrices. The standard form is based on the greatest common factor of subdeterminants of the $\boldsymbol{\lambda}$-matrix. The diagonal elements are ratios of these greatest common factors as defined in Definition 1. We show that that every $\boldsymbol{\lambda}$-matrix will have an unique standard form. We show that two equivalent $\boldsymbol{\lambda}$-matrices will have the same standard form. We extend the Segre characteristic to apply to our standard form. Our standard form is weaker than the canonical form of the single variable case in that that two matrices with the same standard from are not in general equivalent.

DEFINITION 1 (STANDARD FORM) *Let $\mathbf{A}$ be a $\boldsymbol{\lambda}$-matrix. Let p_i be the greatest common monic factor of all i-rowed subdeterminants of $\mathbf{A}$, if not all are zeros, and let $p_i \equiv 0$ if all of them are zero. Let $p_0 = 1$ then the standard form $\mathbf{D}(\boldsymbol{\lambda})$ is*

$$\mathbf{D}(\boldsymbol{\lambda}) = \begin{bmatrix} d_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d_r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & & & & & & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad (1)$$

where r is the maximal integer such that $p_r \neq 0$, and the diagonal elements, d_i are

$$d_i = \frac{p_i}{p_{i-1}}, \quad i = 1, \ldots, r. \qquad (2)$$

(By an i-rowed subdeterminant we mean the determinant of an $i \times i$ submatrix of $\mathbf{A}$.) The proofs of all the results of this subsection can be found in the Appendix.

PROPOSITION 2 *If p is a monic factor of all i-rowed subdeterminants of a $\boldsymbol{\lambda}$-matrix $\mathbf{A}$, it will be a factor of all i-rowed subdeterminants of every $\boldsymbol{\lambda}$-matrix $\mathbf{B}$ which is equivalent to $\mathbf{A}$.*

PROPOSITION 3 *If $\mathbf{A}$ and $\mathbf{B}$ are equivalent $\boldsymbol{\lambda}$-matrices of rank r and p_i is the greatest common factor of the i-rowed subdeterminants ($i \leq r$) of $\mathbf{A}$, then p_i is also the greatest common factor of the i-rowed subdeterminants of $\mathbf{B}$.*

PROPOSITION 4 *If $\mathbf{A}$ and $\mathbf{B}$ are equivalent $\boldsymbol{\lambda}$-matrices then $\mathbf{A}$ and $\mathbf{B}$ will have the same standard form.*

PROPOSITION 5 *If* $\mathbf{A}$ *is a single variable* λ*-matrix then the standard form is the same as the canonical form.*

By taking the d_i of the standard form (see Definition 1) as an extension of the invariant factors of the canonical form, we can extend the Segre characteristic to the multi-variable $\boldsymbol{\lambda}$-matrices. We factor each d_i into irreducible polynomials. We place the powers of the linear factors into the Segre characteristic as before. We may also have factors which are not linear. We define two new symbols, $\mathfrak{b}$ and $\mathfrak{c}$ to represent the first powers of degree two irreducible and degree three irreducible polynomials, respectively. Since we are interested in 3×3 $\boldsymbol{\lambda}$-matrices with polynomials of degree at most 1, we will not have any other non-linear irreducibles. As an example, if our set of factors of the d's is

$$(\lambda_1 + 4\lambda_2 - 8\lambda_3), (\lambda_1^2 + \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3^2)$$

then the extended Segre characteristic is $[\mathfrak{b}\ 1]$.

PROPOSITION 6 *If* $\mathbf{A}$ *and* $\mathbf{B}$ *are equivalent* 3×3 $\boldsymbol{\lambda}$*-matrices of degree 1, then* $\mathbf{A}$ *and* $\mathbf{B}$ *will have the same monic determinant and the same extended Segre characteristic.*

3. FAMILIES OF CONICS AND THEIR INVARIANTS

As is well known, a conic can be represented by the quadratic form

$$S = \mathbf{x}^t \mathbf{A} \mathbf{x} = \sum_{i=1}^{3} \sum_{j=1}^{3} a_{i,j} x_i x_j$$

where $\mathbf{A} = (a_{ij})$ is a real symmetric 3×3 matrix, $\mathbf{x}$ is a vector of homogeneous coordinates of dimension 2. Applying a projective transformation $\mathbf{P}$ to a point $\mathbf{x}$ defines a new point $\mathbf{x}' = \mathbf{P}\mathbf{x}$, where $\mathbf{P}$ is a non-singular 3×3 matrix. A projection $\mathbf{P}$ would change $S = \mathbf{x}^t \mathbf{A} \mathbf{x}$ to $S' = \mathbf{x}^t \mathbf{A}' \mathbf{x}$ where $\mathbf{A}' = \mathbf{P}^t \mathbf{A} \mathbf{P}$. Thus, two conics S_1 and S_2 represented by $\mathbf{A}_1$ and $\mathbf{A}_2$ respectively, are *projectively equivalent* if there is a projection which takes one to the other. In other words, there exists a non-singular matrix $\mathbf{P}$ such that $\mathbf{A}_2 = \mathbf{P}^t \mathbf{A}_1 \mathbf{P}$.

Two families of conics, $\{S_i : 1 \leq i \leq n\}$, $\{T_i : 1 \leq i \leq n\}$ with each conic S_i, T_i being represented by the matrix $\mathbf{A}_i$, $\mathbf{B}_i$ respectively, are called *projectively equivalent* if there is a projection which takes each conic of the first family to the corresponding conic of the second family. In other words, there exists a non-singular matrix $\mathbf{P}$ such that

$$\mathbf{B}_i = \mathbf{P}^t \mathbf{A}_i \mathbf{P}, \quad \text{where} \quad 1 \leq i \leq n$$

A $\boldsymbol{\lambda}$-matrix $\mathbf{A}$ can be associated with a family of coplanar conics as follows:

DEFINITION 2 *The* $\boldsymbol{\lambda}$*-matrix* A *is associated with the family of conics,* $\{S_i : 1 \leq i \leq n\}$ *represented by matrices* $\{\mathbf{A}_i : 1 \leq i \leq n\}$ *if* $\mathbf{A}$ *is a linear combination of the matrices* $\mathbf{A}_i$*.*

$$\mathbf{A} = \sum_{i=1}^{n} \lambda_i \mathbf{A}_i \tag{3}$$

The entries of $\mathbf{A}$ *are polynomials in the indeterminates* $\lambda_1, \ldots, \lambda_n$*. Here,* $\mathbf{A}$ *is a symmetric* 3×3 *matrix since each* $\mathbf{A}_i$ *is a symmetric* 3×3 *matrix.*

Example: Given three coplanar conics (in this case circles), $S_1 \equiv x^2 + y^2 - 1$, $S_2 \equiv 4x^2 + 4y^2 - 1$, and $S_3 \equiv 9x^2 + 9y^2 - 5$, the matrices A_i are

$$\mathbf{A}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \mathbf{A}_2 = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \mathbf{A}_3 = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & -5 \end{bmatrix}$$

and the associated $\boldsymbol{\lambda}$-matrix is

$$\mathbf{A} = \begin{bmatrix} \lambda_1 + 4\lambda_2 + 9\lambda_3 & 0 & 0 \\ 0 & \lambda_1 + 4\lambda_2 + 9\lambda_3 & 0 \\ 0 & 0 & -\lambda_1 - \lambda_2 - 5\lambda_3 \end{bmatrix}$$

The standard form $\mathbf{D}(\lambda)$ of $\mathbf{A}$ is

$$\mathbf{D}(\lambda) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda_1 + 4\lambda_2 + 9\lambda_3 & 0 \\ 0 & 0 & \begin{array}{c} \lambda_1^2 + 5\lambda_1\lambda_2 + 14\lambda_1\lambda_3 + 4\lambda_2^2 \\ +29\lambda_2\lambda_3 + 45\lambda_3^2 \end{array} \end{bmatrix}$$

and the Segre Characteristic is $[(1\ 1)\ 1]$. □

By the results of section II.B, any 3×3 $\boldsymbol{\lambda}$-matrix equivalent to $\mathbf{A}$ must have the same monic determinant and Segre characteristic. The following result will allow us to use the determinant and the Segre characteristic of $\boldsymbol{\lambda}$-matrices as a test for the equivalence of families of conics.

THEOREM 1 *The $\boldsymbol{\lambda}$-matrices associated with different families of coplanar conics are equivalent if and only if the families of conics are projectively equivalent.*

Proof: $\Rightarrow$ Let **A** and **B** be equivalent matrices with **A** and **B** being the $\boldsymbol{\lambda}$-matrices associated with two different families of coplanar conics, $\{\mathbf{A}_i : 1 \leq i \leq n\}$ and $\{\mathbf{B}_i : 1 \leq i \leq n\}$ respectively. Since A and B are equivalent then by Lemma 2 (presented in the appendix) $\mathbf{B} = \mathbf{MAN}$ where **M** and **N** are regular $\boldsymbol{\lambda}$-matrices of degree 0. That is, **M** and **N** are matrices whose entries are only scalars. Replacing **A** and **B** by their defining equations, and distributing **M** and **N** through **A** yields:

$$\sum_{i=1}^{n} \lambda_i \mathbf{B}_i = \sum_{i=1}^{n} \lambda_i \mathbf{M}\mathbf{A}_i\mathbf{N}$$

thus

$$\mathbf{B}_i = \mathbf{M}\mathbf{A}_i\mathbf{N} \quad (1 \leq i \leq n)$$

and since $\mathbf{B}_i$ and $\mathbf{A}_i$ are symmetric matrices, the relation $\mathbf{B}_i = \mathbf{M}\mathbf{A}_i\mathbf{N}$ implies that there exists a non-singular matrix **P** such that $\mathbf{B}_i = \mathbf{P}^t\mathbf{A}_i\mathbf{P}$ [2], Theorem 1, section 102. Furthermore, **P** depends only on **M** and **N**, not $\mathbf{B}_i$ nor $\mathbf{A}_i$. Thus there exists a non-singular matrix **P** such that

$$\mathbf{B}_i = \mathbf{P}^t\mathbf{A}_i\mathbf{P}, \quad (1 \leq i \leq n)$$

Therefore, the families of coplanar conics, $\mathbf{A}_i$ and $\mathbf{B}_i$, are projectively equivalent.
$\Leftarrow$ Since the families of conics $\mathbf{A}_i$ and $\mathbf{B}_i$ are projectively equivalent, there exist a non-singular matrix **P** such that

$$\mathbf{B}_i = \mathbf{P}^t\mathbf{A}_i\mathbf{P}, \quad (1 \leq i \leq n)$$

Substituting into the defining equation for **B** yields

$$\mathbf{B} = \sum_{i=1}^{n} \lambda_i \mathbf{P}^t\mathbf{A}_i\mathbf{P} = \mathbf{P}^t(\sum_{i=1}^{n} \lambda_i \mathbf{A}_i)\mathbf{P} = \mathbf{P}^t\mathbf{A}\mathbf{P}$$

Therefore **A** and **B** are equivalent $\boldsymbol{\lambda}$-matrices. ∎

The main idea of our technique can now be described as follows:

- Using Theorem 3.1, we shall look for invariants over equivalence classes of λ-matrices as these will, by necessity, be invariants of a family of coplanar conics. The invariants we shall use are based on the Segre characteristic and the determinant of the $\boldsymbol{\lambda}$-matrix.

Let **A** be the associated $\boldsymbol{\lambda}$-matrix for a family of conics. Then the determinant of **A** is a homogeneous third degree polynomial

$$|\mathbf{A}| = \Theta_{1,1,1}\lambda_1^3 + \Theta_{1,1,2}\lambda_1^2\lambda_2 + \cdots + \Theta_{n,n,n}\lambda_n^3 \quad (4)$$

where the coefficient of each monomial $\lambda_1^{d_1}\lambda_2^{d_2}\ldots\lambda_n^{d_n}$ is $\Theta_{u,v,w}$, where $\sum_{i=1}^{n} d_i = 3$. The subscripts of $\Theta_{u,v,w}$ are the subscripts of the term, $\lambda_u, \lambda_v\lambda_w$, of which it is the coefficient with $u \leq v \leq w$. The number of Θ's, which we will call q, is a 3-combination of a set of n elements with repetition allowed (see e.g., [26], page 274), The number of Θ's, which we will call q, is a 3-combination of a set of n elements with repetition allowed (see e.g., [26], page 274). Thus,

$$q = \frac{(n+2)!}{6\ (n-1)!} \quad (5)$$

In (4) the Θ's are mixed polynomials in the entries of A_i's (the coefficients of the conics). Each Θ is homogeneous in the coefficients of each conic. Each Θ is an invariant of weight 2 since $|\mathbf{A}| = |\mathbf{P}|^2|\mathbf{A}|$ for any projection matrix **P** which acts on the family of conics. Following the method found in [30], page 171, a polynomial Φ in the Θ's will be an *invariant* if the following conditions hold:

- *Condition 1.* Φ is homogeneous, of degree, say, r, in the Θ's
- *Condition 2.* Φ is homogeneous in the coefficients of each conic. The polynomial Φ is homogeneous in the coefficients of the i^{th} conic if the sum in (7) is constant for all monomials of Φ. A monomial of Φ has the form:

$$\prod_{j=1}^{q} \Theta_{u_j,v_j,w_j}^{k_j} \ (\text{where } \sum_{j=1}^{q} k_j = r) \quad (6)$$

which has the following weight in the coefficients of the i^{th} conic :

$$\sum_{j=0}^{q} k_j(\delta_{v_j} + \delta_{u_j} + \delta_{w_j}) \quad (7)$$

where

$$\delta_x = \begin{cases} 1 & textif\ x = i, \\ 0 & otherwise \end{cases} \quad \text{for } x = u_j, v_j, \text{and } w_j \tag{8}$$

The invariant Φ will have weight $2r$ where r is the degree in condition 1. An absolute invariant can be built from two invariant polynomials Φ and Ψ of equal weight by taking the rational function Φ/Ψ. It is also desirable, but not required, that for each conic, the sum in condition 2 is the same for both Φ and Ψ. If the sum in condition 2 is the same for both Φ and Ψ, then the invariant is not affected by scalar multiples of the individual matrices representing the conics. This is desirable since any matrix $k\mathbf{A}_i$ for any nonzero scalar k, will still represent the same i^{th} conic. By having the same weight in the coefficients of the i^{th} conic for both Φ and Ψ will cause the scalar k to cancel out.

Table 1. Independent Absolute Invariants of Families of Conics for small number of conics.

Number of Conics	Independent absolute invariants
2	$\mathcal{I}_1^2 = \frac{\Theta_{1,1,1}\Theta_{1,2,2}}{\Theta_{1,1,2}^2}$, $\mathcal{I}_2^2 = \frac{\Theta_{2,2,2}\Theta_{1,1,2}}{\Theta_{1,2,2}^2}$
3	$\mathcal{I}_1^3 = \frac{\Theta_{1,1,1}\Theta_{1,2,2}\Theta_{2,2,3}\Theta_{3,3,3}}{\Theta_{1,1,2}^2\Theta_{2,3,3}^2}$, $\mathcal{I}_2^3 = \frac{\Theta_{2,2,2}\Theta_{1,1,2}\Theta_{1,1,3}\Theta_{3,3,3}}{\Theta_{1,2,2}^2\Theta_{1,3,3}^2}$, $\mathcal{I}_3^3 = \frac{\Theta_{1,1,1}\Theta_{1,3,3}\Theta_{2,3,3}\Theta_{2,2,2}}{\Theta_{1,1,3}^2\Theta_{2,2,3}^2}$, $\mathcal{I}_4^3 = \frac{\Theta_{1,1,2}\Theta_{1,1,3}\Theta_{2,2,3}\Theta_{2,3,3}}{\Theta_{1,2,2}\Theta_{1,2,3}^2\Theta_{1,3,3}}$, $\mathcal{I}_5^3 = \frac{\Theta_{1,1,2}\Theta_{1,2,2}\Theta_{1,3,3}\Theta_{2,3,3}}{\Theta_{1,1,1}\Theta_{1,2,3}\Theta_{2,2,2}\Theta_{3,3,3}}$, $\mathcal{I}_6^3 = \frac{\Theta_{1,1,3}\Theta_{1,2,2}\Theta_{1,3,3}\Theta_{2,2,3}}{\Theta_{1,1,2}\Theta_{1,2,3}^2\Theta_{2,3,3}}$, $\mathcal{I}_7^3 = \frac{\Theta_{1,1,1}\Theta_{1,1,3}\Theta_{1,3,3}\Theta_{2,2,2}\Theta_{2,2,3}\Theta_{2,3,3}}{\Theta_{1,1,2}^2\Theta_{1,2,2}^2\Theta_{3,3,3}^2}$
4	$\mathcal{I}_1^4 = \frac{\Theta_{1,1,1}\Theta_{1,1,2}\Theta_{2,3,4}}{\Theta_{1,1,3}\Theta_{1,1,4}\Theta_{1,2,2}}$, $\mathcal{I}_2^4 = \frac{\Theta_{2,2,2}\Theta_{1,2,2}\Theta_{1,2,4}}{\Theta_{2,2,3}\Theta_{2,2,4}\Theta_{1,1,2}}$, $\mathcal{I}_3^4 = \frac{\Theta_{3,3,3}\Theta_{1,3,3}\Theta_{1,2,4}}{\Theta_{2,3,3}\Theta_{3,3,4}\Theta_{1,1,3}}$, $\mathcal{I}_4^4 = \frac{\Theta_{4,4,4}\Theta_{1,4,4}\Theta_{1,2,3}}{\Theta_{2,4,4}\Theta_{3,4,4}\Theta_{1,1,4}}$, $\mathcal{I}_5^4 = \frac{\Theta_{1,1,1}\Theta_{1,1,3}\Theta_{2,3,4}}{\Theta_{1,1,2}\Theta_{1,1,4}\Theta_{1,3,3}}$, $\mathcal{I}_6^4 = \frac{\Theta_{1,1,1}\Theta_{1,1,4}\Theta_{2,3,4}}{\Theta_{1,1,2}\Theta_{1,1,3}\Theta_{1,4,4}}$, $\mathcal{I}_7^4 = \frac{\Theta_{3,3,3}\Theta_{2,3,3}\Theta_{1,2,4}}{\Theta_{1,3,3}\Theta_{3,3,4}\Theta_{2,2,3}}$, $\mathcal{I}_8^4 = \frac{\Theta_{4,4,4}\Theta_{2,4,4}\Theta_{1,2,3}}{\Theta_{1,4,4}\Theta_{3,4,4}\Theta_{2,2,4}}$, $\mathcal{I}_9^4 = \frac{\Theta_{2,2,2}\Theta_{2,2,3}\Theta_{1,3,4}}{\Theta_{1,3,3}\Theta_{3,3,4}\Theta_{2,3,3}}$, $\mathcal{I}_{10}^4 = \frac{\Theta_{2,2,2}\Theta_{2,2,4}\Theta_{1,3,4}}{\Theta_{1,2,2}\Theta_{2,2,3}\Theta_{2,4,4}}$, $\mathcal{I}_{11}^4 = \frac{\Theta_{3,3,3}\Theta_{1,2,4}\Theta_{3,3,4}}{\Theta_{1,3,3}\Theta_{2,3,3}\Theta_{3,4,4}}$, $\mathcal{I}_{12}^4 = \frac{\Theta_{4,4,4}\Theta_{3,4,4}\Theta_{1,2,3}}{\Theta_{1,4,4}\Theta_{2,4,4}\Theta_{3,3,4}}$

The number of independent absolute invariants of a family of conics depends on the number of conics in the family. The number of independent absolute invariants may be found by using dimensional analysis as done in [11]. For two conics there are two independent absolute invariants. For three conics there are seven independent absolute invariants. Each additional conic will add five more independent absolute invariants. The absolute invariants are not unique. Any set of them can be used. Depending on the application, we may not want to use all of the independent absolute invariants, but only a subset of the independent absolute invariants. When using a subset, we make sure that the sum in condition 2 for a conic is not zero for all of the invariants in the subset; if it is zero then the invariants in that subset will not depend on that conic.

A set of these independent absolute invariants and the Segre characteristic will be used as the invariant descriptors of a family of coplanar conics. We will call our absolute invariants $\mathcal{I}_j^n$ which means the j^{th} absolute invariant of a family containing n conics. A set of invariants for three and four conic families are presented in Table 1. Example invariants for five, six, and seven conic families are located in Table 11. It should be pointed out that the Segre Characteristic is not a continuous invariant. It partitions the families of coplanar conics into a small number of broad classes. At the boundaries of these classes, the Segre Characteristic can be changed by an arbitrary small change of the parameters defining the conics.

Table 2. Example Absolute Invariants of Families of Conics for large number of conics.

Number of Conics	Absolute invariants
5	$\mathcal{I}_1^5 = \frac{\Theta_{1,1,1}\Theta_{3,3,3}\Theta_{5,5,5}\Theta_{1,2,2}\Theta_{2,2,3}\Theta_{3,4,4}}{\Theta_{3,3,4}\Theta_{2,3,3}\Theta_{1,3,5}\Theta_{1,2,4}\Theta_{1,1,5}\Theta_{2,2,5}}$ $\mathcal{I}_2^5 = \frac{\Theta_{1,1,2}\Theta_{1,1,3}\Theta_{3,5,5}\Theta_{4,5,5}}{\Theta_{1,2,3}\Theta_{1,1,1}\Theta_{3,4,5}\Theta_{5,5,5}}$
6	$\mathcal{I}_1^6 = \frac{\Theta_{1,3,5}\Theta_{2,4,6}\Theta_{2,2,2}\Theta_{5,5,5}\Theta_{1,3,3}\Theta_{4,4,6}}{\Theta_{1,2,3}\Theta_{4,5,6}\Theta_{2,3,4}\Theta_{1,5,6}\Theta_{2,2,3}\Theta_{4,5,5}}$
7	$\mathcal{I}_1^7 = \frac{\Theta_{1,1,2}\Theta_{2,4,6}\Theta_{6,7,7}\Theta_{3,3,3}\Theta_{4,4,4}}{\Theta_{3,3,7}\Theta_{1,5,5}\Theta_{2,2,3}\Theta_{5,6,6}\Theta_{1,4,7}}$

4. COMPARISON WITH PREVIOUS INVARIANTS OF TWO COPLANAR CONICS.

The use of invariants for two coplanar conics have been proposed by a number of authors. In this section we compare the use of our invariant descriptors with those previously used. The comparison will be made against invariants presented by Forsyth [8], Quan [24], and Maybank [17].

For two coplanar conics the invariant descriptors that we propose using conics are $\mathcal{I}_1^2$, $\mathcal{I}_2^2$, and the Segre characteristic. The invariants $\mathcal{I}_1^2$ and $\mathcal{I}_2^2$ are the absolute invariants given in Table 1 and given again in this section in (11). Let $\mathbf{A}_1$ and $\mathbf{A}_2$ be the coefficient matrices of two conics, the λ-matrix $\mathbf{A}$ is

$$\mathbf{A} = \lambda_1 \mathbf{A}_1 + \lambda_2 \mathbf{A}_2$$

which can be restated as a single variable λ-matrix with a change of variables if we add the further restriction that $\mathbf{A}_1$ is nonsingular. We will do so that we may use the stronger results from section2.2.1. Namely, we can use the standard algorithm [13], pp. 89-91 for diagonalizing the single variable λ-matrix, and we can use Proposition 1 to prove the completeness of our invariants. The λ-matrix $\mathbf{A}$ is now

$$\mathbf{A} = \mathbf{A}_1 + \lambda \mathbf{A}_2 \tag{9}$$

The determinant of $\mathbf{A}$ given by (9) is

$$|\mathbf{A}| = \Theta_{3,0} + \Theta_{2,1}\lambda + \Theta_{1,2}\lambda^2 + \Theta_{0,3}\lambda^3 \tag{10}$$

where the notation for Θ's are as explained just after (4). The invariants $\mathcal{I}_1^2$ and $\mathcal{I}_2^2$ are

$$\mathcal{I}_1^2 = \frac{\Theta_{3,0}\Theta_{1,2}}{\Theta_{2,1}^2}, \ \mathcal{I}_2^2 = \frac{\Theta_{0,3}\Theta_{2,1}}{\Theta_{1,2}^2} \tag{11}$$

The invariants α and β presented in [24] are the same independent absolute invariants $\mathcal{I}_1^2$ and $\mathcal{I}_2^2$ that we are also using. The invariants I_1 and I_2 presented in [8] are

$$I_1 = Trace\left[\mathbf{A}_1^{-1}\mathbf{A}_2\right], \ \ I_2 = Trace\left[\mathbf{A}_2^{-1}\mathbf{A}_1\right]$$

with the additional constraint that the matrices $\mathbf{A}_1$ and $\mathbf{A}_2$ are normalized by setting their determinants to one. These two invariants can be expressed as a function of the Θ's without normalizing the matrices $\mathbf{A}_1$ and $\mathbf{A}_2$:

$$I_1 = \left(\frac{\Theta_{2,1}^3}{\Theta_{3,0}^2\Theta_{0,3}}\right)^{\frac{1}{3}}, \ I_2 = \left(\frac{\Theta_{1,2}^3}{\Theta_{0,3}^2\Theta_{3,0}}\right)^{\frac{1}{3}}$$

if the matrices $\mathbf{A}_1$ and $\mathbf{A}_2$ are all ready normalized then $I_1 = \Theta_{2,1}$ and $I_2 = \Theta_{1,2}$.

The invariants presented by Maybank [17] are the *t-invariants* of the pair of conics. They are a function of the roots of (10). Letting μ_1,μ_2,μ_3 be the roots of (10), then the t-invariants are

$$t_1 = \frac{\mu_1\mu_2 + \mu_2\mu_3 + \mu_1\mu_3}{(\mu_1 + \mu_2 + \mu_3)^2}, \ t_2 = \frac{\mu_1\mu_2\mu_3}{(\mu_1 + \mu_2 + \mu_3)^3}$$

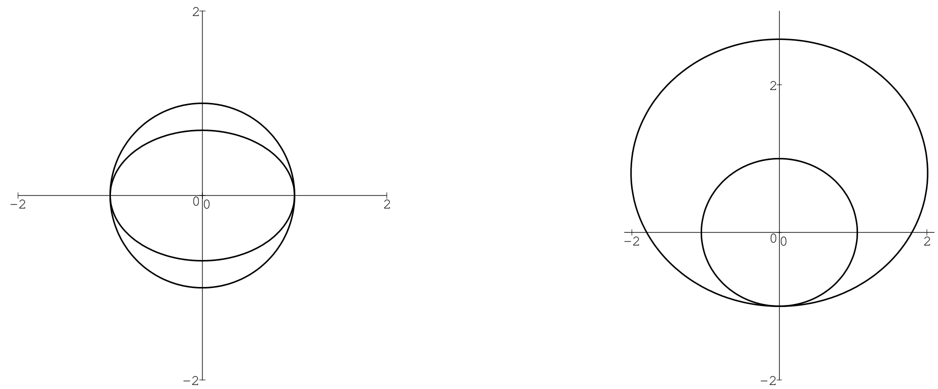
and they can be expressed in terms of the Θ's as

$$t_1 = \frac{\Theta_{2,1}\Theta_{0,3}}{\Theta_{1,2}^2} = \mathcal{I}_2^2, \ t_2 = \frac{\Theta_{3,0}\Theta_{0,3}{}^2}{\Theta_{1,2}^3}$$

All of the above invariants can be expressed in terms of the Θ's. By combining Theorem 1 and Proposition 1, we see that using just the Θ's are not enough to show that two pairs of coplanar conics are equivalent. The additional information of the Segre characteristic is needed to show the two pairs of coplanar conics are equivalent. To demonstrate this, we now present two pairs of conics in Figure 1. For these two families of conics, just using the Θ's is not enough to distinguish between them. The use of the Segre characteristic discriminates between the two pairs.

The pair of conics in Figure 1(a) is $x^2 + 2y^2 - 1 = 0$ and $x^2 + y^2 - 1 = 0$. The pair of conics in Figure 1(b) is $x^2 + \sqrt{5}y^2 - \sqrt{5} = 0$ and $x^2 + (\sqrt{5} - 1)y^2 - 2y - 1 - \sqrt{5} = 0$ which is detected as the pair $x^2 + 2.236y^2 - 2.236 = 0$ and $x^2 + 1.236y^2 - 2y - 3.236 = 0$. The invariants are calculated and presented in Table 3.

In Table 3 the conics of Figure 1(a) and the conics of Figure 1(b) are not projectively related. This is obvious since the first family has double contact (two points of intersection, both multiplicity two) and the second family has a single contact of the first order (three points of intersection, one of which is multiplicity two). The invariants of [8], [24], and [17] give the same value for both families of conics. The addition of the Segre characteristic shows the two families are not projectively related.



(a) (b)

Fig. 1. Pairs of coplanar conics.

Table 3. Comparison of Invariants for Conic Pairs in Figure 1.

	Forsyth		Maybank		Quan		Segre
	I_1	I_2	t_1	t_2	$\mathcal{I}_1^2$	$\mathcal{I}_2^2$	Characteristic
Figure 1(a)	3.150	3.175	0.3125	0.03125	0.3200	0.3125	[(1 1) 1]
Figure 1(b)	3.150	3.175	0.3125	0.03125	0.3200	0.3125	[2 1]

A complete listing of the relationship of the Segre Characteristic and the intersection properties of a pair of conics may be found in [2], page 309.

5. EXPERIMENTAL RESULTS

The results were processed on a Silicon Graphics Indy Workstation. The software used was *AVS* version 5.2 [1], *Maple* version V R3 [16], and *Python* version 1.2 [21]. Maple is an algebraic package and was used to calculate the invariants of the conics found in the images. Python was used as a scripting language to control AVS through its remote command line interface and to pass information to Maple. Edge detection and thresholding modules from AVS were used to preprocess the images. The ellipses in the images were detected by using a variation of the Hough transform as presented in [12]. The coefficients of a ellipse were calculated by applying the Brookstein algorithm (see e.g. [27], page 246) to the edge points corresponding to the detected ellipse. We assume that the correspondence of conics between images is known. We do this by selecting the corresponding ellipses by hand from all of the ellipses fitted in an image. First we investigate the affects of noise on the invariants. Then we apply the three and four conic invariants to match objects. Lastly, we use six and seven conic invariants to recognize tracked vehicles.

5.1. *Comparing values of the Invariants under Noise*

In order to compare two values of an invariant, a distance measure is needed. Just using the value of the invariant directly does not give a good partitioning of the families of ellipses that generate the invariants. We instead use the probability distribution to compare the values similarly to the methods presented in [18] and [27]. We define a distance between two values of an invariant, v_1 and v_2, to be the probability that a value v of the

Table 4. Noise affects on the Invariants. This Table presents the average distance of the invariants between a family of noised ellipses and the original family of ellipses. Noise is added to the major axis, minor axis, rotation, and center point of the original ellipses.

	Amount of noised added to the Original Ellipse						
Invariant	0.1%	1%	2%	3%	5%	10%	25%
$\mathcal{I}_1^2$	0.0004	0.0038	0.0076	0.0113	0.0186	0.0360	0.0816
$\mathcal{I}_2^2$	0.0004	0.0037	0.0074	0.0111	0.0183	0.0356	0.0804
I_1	0.0005	0.0055	0.0108	0.0158	0.0251	0.0462	0.0949
I_2	0.0006	0.0056	0.0107	0.0155	0.0247	0.0451	0.0919
t_1	0.0004	0.0037	0.0074	0.0111	0.0183	0.0356	0.0804
t_2	0.0006	0.0063	0.0121	0.0174	0.0267	0.0430	0.0610
$\mathcal{I}_1^3$	0.0010	0.0091	0.0172	0.0246	0.0384	0.0661	0.1243
$\mathcal{I}_2^3$	0.0010	0.0093	0.0174	0.0246	0.0381	0.0655	0.1226
$\mathcal{I}_3^3$	0.0010	0.0101	0.0181	0.0256	0.0393	0.0682	0.1247
$\mathcal{I}_4^3$	0.0014	0.0137	0.0257	0.0364	0.0555	0.0910	0.1545
$\mathcal{I}_5^3$	0.0010	0.0105	0.0200	0.0291	0.0460	0.0801	0.1450
$\mathcal{I}_6^3$	0.0014	0.0141	0.0261	0.0371	0.0563	0.0925	0.1544
$\mathcal{I}_7^3$	0.0014	0.0124	0.0229	0.0323	0.0498	0.0849	0.1497
$\mathcal{I}_1^4$	0.0011	0.0103	0.0200	0.0294	0.0461	0.0809	0.1514
$\mathcal{I}_2^4$	0.0010	0.0101	0.0200	0.0295	0.0469	0.0827	0.1526
$\mathcal{I}_3^4$	0.0011	0.0105	0.0206	0.0298	0.0472	0.0830	0.1505
$\mathcal{I}_4^4$	0.0011	0.0109	0.0208	0.0299	0.0463	0.0825	0.1486
$\mathcal{I}_5^4$	0.0012	0.0107	0.0203	0.0292	0.0462	0.0812	0.1508
$\mathcal{I}_6^4$	0.0010	0.0103	0.0197	0.0289	0.0454	0.0808	0.1513
$\mathcal{I}_7^4$	0.0012	0.0113	0.0213	0.0306	0.0478	0.0831	0.1515
$\mathcal{I}_8^4$	0.0010	0.0107	0.0204	0.0294	0.0464	0.0821	0.1503
$\mathcal{I}_9^4$	0.0011	0.0103	0.0196	0.0281	0.0441	0.0763	0.1310
$\mathcal{I}_{10}^4$	0.0012	0.0102	0.0190	0.0270	0.0424	0.0729	0.1247
$\mathcal{I}_{11}^4$	0.0011	0.0111	0.0215	0.0309	0.0485	0.0842	0.1527
$\mathcal{I}_{12}^4$	0.0010	0.0109	0.0209	0.0299	0.0465	0.0825	0.1496
$\mathcal{I}_1^5$	0.0016	0.0160	0.0294	0.0410	0.0626	0.1006	0.1580
$\mathcal{I}_2^5$	0.0013	0.0122	0.0234	0.0335	0.0530	0.0943	0.1692
$\mathcal{I}_1^6$	0.0016	0.0161	0.0299	0.0414	0.0629	0.1036	0.1620
$\mathcal{I}_1^7$	0.0014	0.0148	0.0280	0.0397	0.0606	0.1024	0.1641

invariant lies in the interval $[v_1, v_2]$. A cumulative probability distribution function F is created from invariants calculated from random generated ellipses. Taking into account the similarity of our invariants with large absolute values, the distance between two invariants v_1 and v_2 is

$$\text{distance}(v_1, v_2) = \min(|F(v_1) - F(v_2)|, \\ 1 - |F(v_1) - F(v_2)|) \quad (12)$$

We generated the ellipses by randomly generating the components (major axis, minor axis, center point, and rotation) such that the ellipse will lie on an 512×512 image. For each invariant, 10,000 values were calculated from randomly generated ellipses. The values were sorted and used to build a cumulative probability distribution for each invariant.

With the randomly generated invariants, the distance measure for two values v_1 and v_2 is just the minimum number of values that lie between v_1 and v_2 when the sort list of values is viewed as a circular array. We will use this distance measure when comparing the invariants calculated from images.

The noise in an image may affect the coefficients of the conics that are found in the image. How the noise affects the coefficients of a conic is dependent on the method used to find and fit the conic. To isolate the effects of noise on the invariants, we add simulated noise to ellipses and compare the invariants of the noisy ellipse and the original ellipse. Instead of adding noise to the coefficients of the conic, we added an percentage of error to the lengths of the major and minor axis,

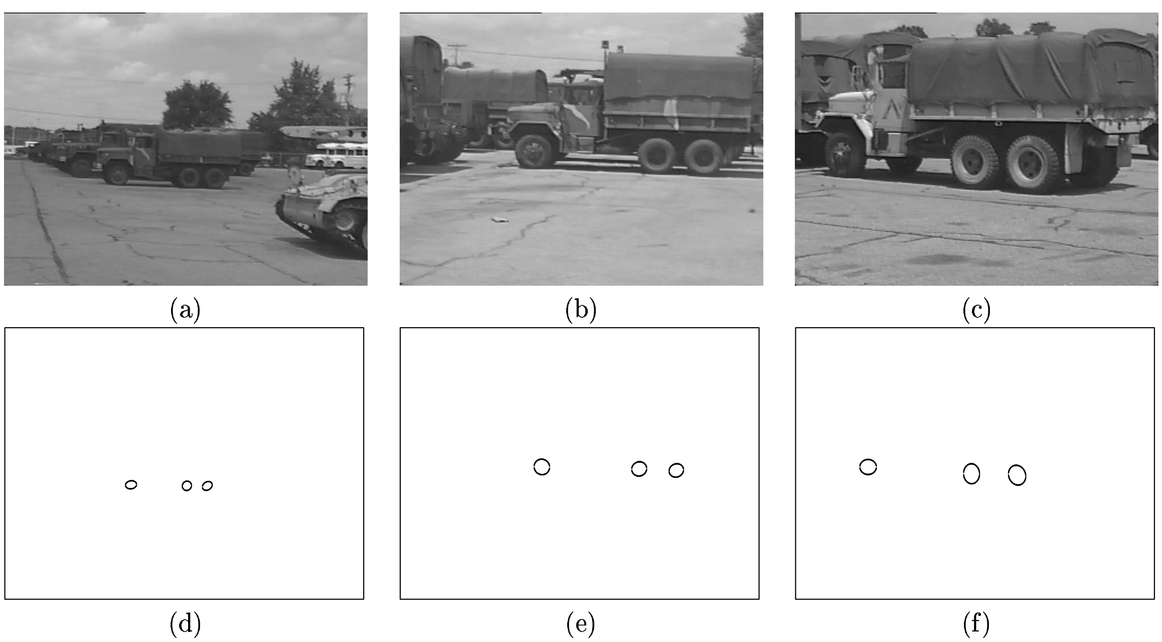
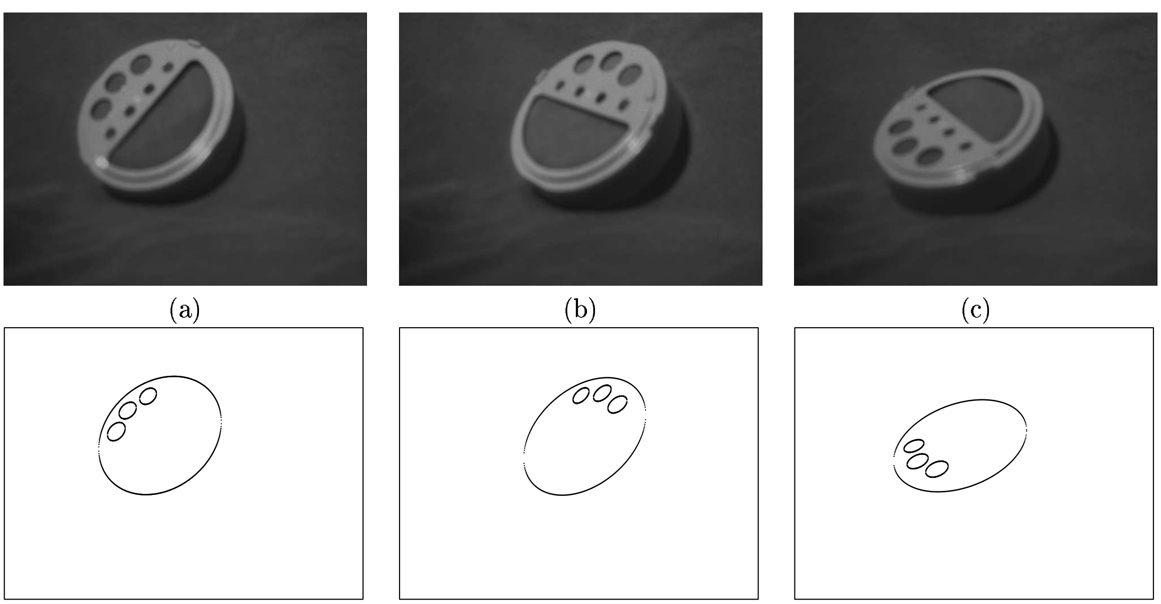


Fig. 2. Trucks



(a) (b) (c)

(j) (k) (l)

Fig. 3. Views of a Shaker Lid

Table 5. Three conic invariants of the tires of the trucks in Figure 2 and of the large inner circles of the lids in Figure 3.

Image	$\mathcal{I}_1^3$	$\mathcal{I}_2^3$	$\mathcal{I}_3^3$	$\mathcal{I}_4^3$	$\mathcal{I}_5^3$	$\mathcal{I}_6^3$	$\mathcal{I}_7^3$
2(a)	0.270e-3	0.834e-4	0.238e-3	0.424e-2	-0.153e6	0.304	0.0599
2(b)	0.283e-3	0.137e-4	0.181e-3	0.150e-2	-0.401e6	0.384	0.0811
2(c)	0.112e-3	0.607e-4	0.811e-4	0.838e-2	-0.443e6	0.322	0.108
3(a)	0.0876	0.0137	0.0166	0.0107	-28.3	0.446	18.8
3(b)	0.0327	0.726e-2	0.613e-2	0.0166	-111	0.441	16.8
3(c)	0.0621	0.931e-2	0.0114	0.0121	-49.7	0.460	21.6

Table 6. Distance between invariants of Table 5.

		2(a)	2(b)	2(c)	3(a)	3(b)	3(c)
$\mathcal{I}_1^3$	2(a)	0.0000	0.0008	0.0209	0.2888	0.3693	0.2677
	2(b)	0.0008	0.0000	0.0217	0.2880	0.3685	0.2669
	2(c)	0.0209	0.0217	0.0000	0.3097	0.3902	0.2886
	3(a)	0.2888	0.2880	0.3097	0.0000	0.0805	0.0211
	3(b)	0.3693	0.3685	0.3902	0.0805	0.0000	0.1016
	3(c)	0.2677	0.2669	0.2886	0.0211	0.1016	0.0000
$\mathcal{I}_2^3$	2(a)	0.0000	0.0222	0.0064	0.2035	0.1672	0.1793
	2(b)	0.0222	0.0000	0.0158	0.2257	0.1894	0.2015
	2(c)	0.0064	0.0158	0.0000	0.2099	0.1736	0.1857
	3(a)	0.2035	0.2257	0.2099	0.0000	0.0363	0.0242
	3(b)	0.1672	0.1894	0.1736	0.0363	0.0000	0.0121
	3(c)	0.1793	0.2015	0.1857	0.0242	0.0121	0.0000
$\mathcal{I}_3^3$	2(a)	0.0000	0.0064	0.0229	0.1987	0.1383	0.1776
	2(b)	0.0064	0.0000	0.0165	0.2051	0.1447	0.1840
	2(c)	0.0229	0.0165	0.0000	0.2216	0.1612	0.2005
	3(a)	0.1987	0.2051	0.2216	0.0000	0.0604	0.0211
	3(b)	0.1383	0.1447	0.1612	0.0604	0.0000	0.0393
	3(c)	0.1776	0.1840	0.2005	0.0211	0.0393	0.0000
$\mathcal{I}_4^3$	2(a)	0.0000	0.0425	0.0383	0.0552	0.0841	0.0616
	2(b)	0.0425	0.0000	0.0808	0.0977	0.1266	0.1041
	2(c)	0.0383	0.0808	0.0000	0.0169	0.0458	0.0233
	3(a)	0.0552	0.0977	0.0169	0.0000	0.0289	0.0064
	3(b)	0.0841	0.1266	0.0458	0.0289	0.0000	0.0225
	3(c)	0.0616	0.1041	0.0233	0.0064	0.0225	0.0000
$\mathcal{I}_5^3$	2(a)	0.0000	0.0023	0.0025	0.2797	0.1919	0.2436
	2(b)	0.0023	0.0000	0.0002	0.2820	0.1942	0.2459
	2(c)	0.0025	0.0002	0.0000	0.2822	0.1944	0.2461
	3(a)	0.2797	0.2820	0.2822	0.0000	0.0878	0.0361
	3(b)	0.1919	0.1942	0.1944	0.0878	0.0000	0.0517
	3(c)	0.2436	0.2459	0.2461	0.0361	0.0517	0.0000
$\mathcal{I}_6^3$	2(a)	0.0000	0.0228	0.0051	0.0359	0.0347	0.0382
	2(b)	0.0228	0.0000	0.0177	0.0131	0.0119	0.0154
	2(c)	0.0051	0.0177	0.0000	0.0308	0.0296	0.0331
	3(a)	0.0359	0.0131	0.0308	0.0000	0.0012	0.0023
	3(b)	0.0347	0.0119	0.0296	0.0012	0.0000	0.0035
	3(c)	0.0382	0.0154	0.0331	0.0023	0.0035	0.0000
$\mathcal{I}_7^3$	2(a)	0.0000	0.0123	0.0257	0.2539	0.2491	0.2591
	2(b)	0.0123	0.0000	0.0134	0.2416	0.2368	0.2468
	2(c)	0.0257	0.0134	0.0000	0.2282	0.2234	0.2334
	3(a)	0.2539	0.2416	0.2282	0.0000	0.0048	0.0052
	3(b)	0.2491	0.2368	0.2234	0.0048	0.0000	0.0100
	3(c)	0.2591	0.2468	0.2334	0.0052	0.0100	0.0000

to the location of the center point, and to the angle of rotation. This resulted in a realistic error in the fitting of a conic. A series of ellipses were randomly generated and the true invariants were calculated. Noise was then added to each axis by randomly varying the axis length in the range of $\pm$ error percentage. The location of the center point was randomly varied by $\pm$ error percentage of 50 pixels. The angle of rotation was randomly varied by $\pm$ error percentage of $\pi/2$. The invariants of the noisy ellipse were calculated and the distance measure from the original was recorded. The process was conducted for $10,000$ ellipses (different from those used to generate the PDFs). The average distance between the noisy ellipses and the original ellipses is presented in Table 4. The columns are the percentage of noise added to the ellipses. The rows are the invariants from Table 1, Table 11 or from section IV (i.e., I_1, I_2, t_1, and t_2).

The invariants behave reasonably well. Especially when considering that the fitting method for conics will remove some of the noise in an image in the same way that the error in fitting a line is less than error of the individual points (for large number of points and random noise). The Table 4 will be used to decide when two families of detected ellipses match by checking the distance measure of the two families against the average distance under a set amount of noise.

5.2. Matching of Three and Four Coplanar Conics

In Figure 2 we present three images of trucks and the conics detected from their wheels. The conics were detected at the edge of the rim of the tires. The tires may not be exactly coplanar especially as the front tire may be turned a little bit, but it is a good approximation in most cases. The three conic invariants were calculated from the detected conics and placed in Table 5. The discriminating power of the invariants is demonstrated by using Figure 3 and focusing on the subset consisting of the three small conics. The detected conics of the shaker lid in images 3(a) through 3(c) is displayed in images 3(d) through 3(f), respectively. The three conics invariants were calculated and placed in Table 5. The distances between the invariants of Figures 2(b), 2(c), 3(a), and 3(c) are collected in Table 6. Thus the three conic invariants may be used to discriminate between different objects.

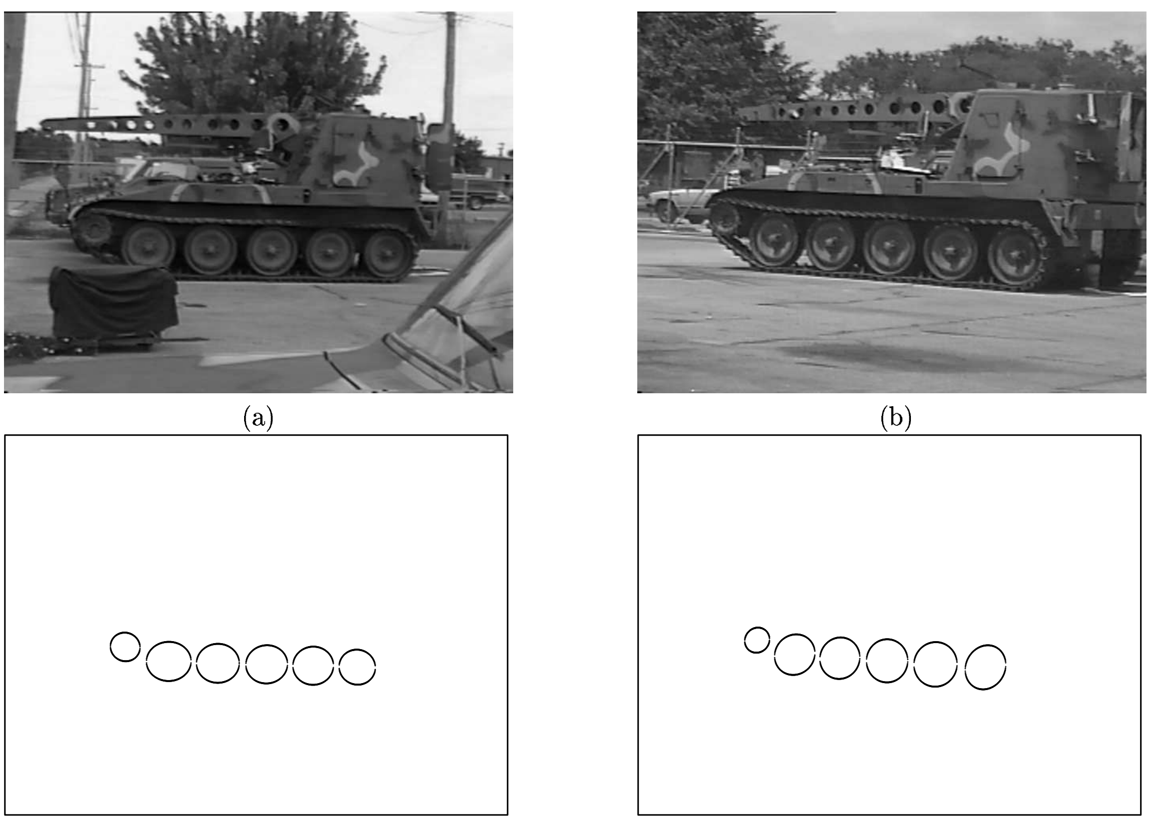
The Segre characteristic is the same for both objects as the conics in both families are disjoint, non-concentric circles. As all of our test objects contain families of disjoint, non-concentric circles, their Segre characteristic is the same. Therefore we do not present the Segre characteristic in the tables of invariants.

The four conic invariants were calculated from the detected conics in Figure 3 and placed in Table 7.

5.3. Recognizing Tracked Vehicles

In this section, we attempt to recognize tracked vehicles based on their wheel configuration. Two images of a crane are presented in Figure 4(a) and 4(b). The conics detected at the wheels of the cranes are presented in Figure 4(c) and 4(d) respectively. The crane has six wheels and so we used the $\mathcal{I}_1^6$ invariant from Table 11. The $\mathcal{I}_1^6$ calculated from the detected conics in Figure 4 can be found in Table 8.

Two images of an ammunition carrier are presented in Figure 5(a) and 5(b). The conics detected at the wheels of the carriers are presented in Figure 5(c) and 5(d) respectively. The invariant $\mathcal{I}_1^7$ of the seven conics of the carriers are presented in Table 8. To show a comparison with the cranes, the first six wheels (starting from the front) of the ammunition carrier were used to calculated the six conic invariant $\mathcal{I}_1^6$. This may also be found Table 8. The values of $\mathcal{I}_1^6$ for the cranes and the carriers are close since the wheels have a similar layout, but there is enough of a difference that they may be used to discriminate between the two object. In Figure 6(a), both a crane and an ammunition carrier is present. The conics detected at the wheels of the two vehicles is displayed in Figure 6(b). The values of $\mathcal{I}_1^6$ and $\mathcal{I}_1^7$ are also presented in Table 8. To recognize to vehicles in Figure 6(a), the distance from the $\mathcal{I}_1^6$ of each vehicle to the $\mathcal{I}_1^6$ value from the Figures 4(a), 4(b), 5(a), and 5(b) were calculated and are presented in Table 9. This al-



(a) (b)

(c) (d)

Fig. 4. Cranes

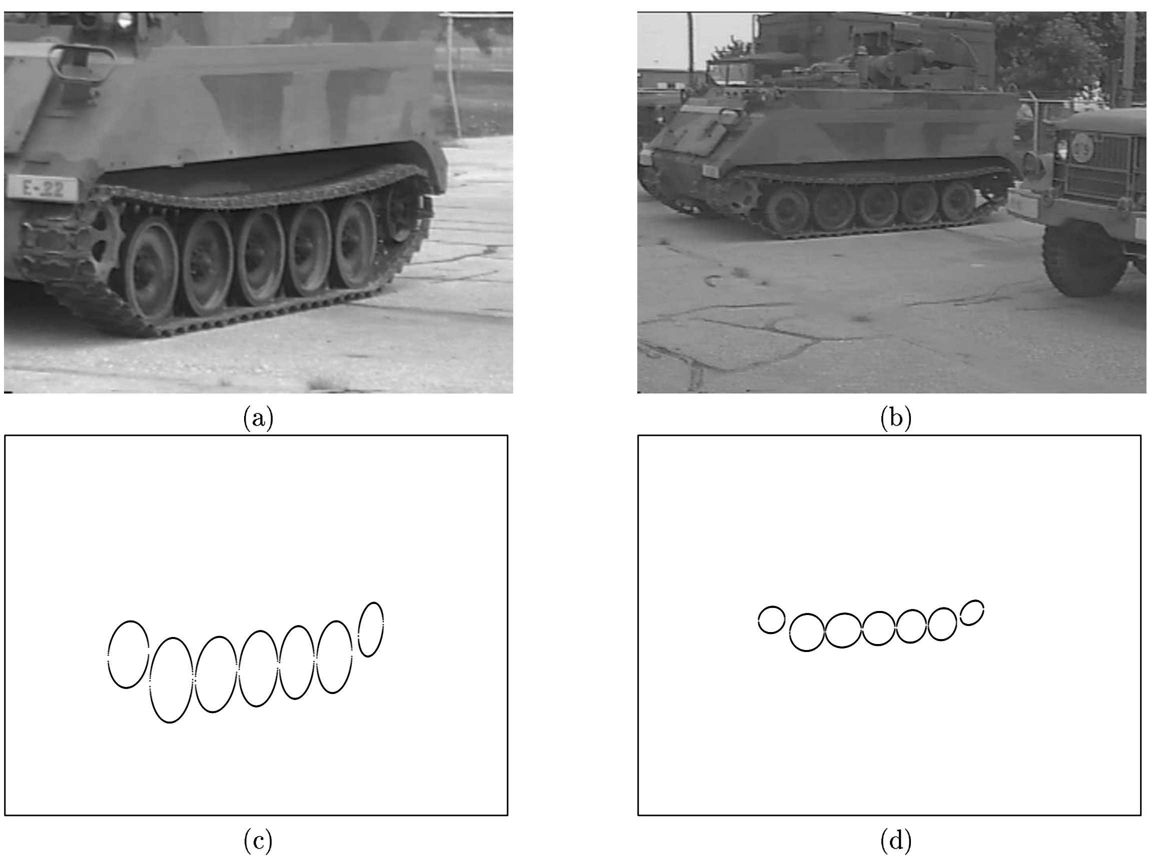


Fig. 5. Ammunition Carrier

(a) (b)

Fig. 6. Multiple Vehicles

lows us to classify the first object as a crane and the second object as an ammunition carrier.

6. SUMMARY

We have extended the use of invariants from recognizing a pair of conics to recognizing any number of conics. We have shown a case where the Segre characteristic can discriminate between pairs of conics while the previously used invariants are not able to make the discrimination. We found that, on average, the invariants respond reasonable well to noise. The invariants were shown to be able match and discriminate between real world objects.

Some of the directions for future research are:

- Find and test symmetric invariants. That is, invariants that do not depend on the order of the conics.
- Find an optimal set of invariants with respect to their stability under noise.
- Extend the results from families of conics to families of Quadrics.
- Find geometric interpretation for the invariants.
- Extend to families of non-coplanar conics following Quan's method in [22], [23].
- Replace the computationally expensive calculation of the Segre Characteristic with a set of inequality checks as was done in [32] for two conics using a Hessian, Jacobian, and a Discriminant.

Appendix

In this appendix we give the proofs of certain results that were stated in section 2.

Lemma 1 *Two 3×3 single variable λ-matrices are equivalent if and only if they have the same monic determinant and the same E_3.*

Proof $\Rightarrow$ Two equivalent single variable λ-matrices will have the same invariant factors. Thus the same E_3 and monic determinant.

Proof $\Leftarrow$ The degree of the determinant Δ is at most 3. The degree of E_3 is at most 3. The linear factors of E_3 and Δ are the same. We distinguish three cases:

- If E_3 is of degree 3 then E_1 and E_2 are 1,
- If E_3 is of degree 2 then E_1 is 1 and E_2 is $\frac{\Delta}{E_3}$.
- If E_3 is of degree 1 then E_1 and E_2 are equal to E_3

Thus in all cases, both the λ-matrices will have the same invariant factors and are thus equivalent. □

Proposition 1 *Two 3×3 single variable λ-matrices are equivalent if and only if the have the same monic determinant and the same Segre characteristic.*

Proof: $\Rightarrow$ Two equivalent single variable λ-matrices will have the same elementary divisors. Thus the same Segre characteristic and monic determinant.

$\Leftarrow$ The degree of the determinant Δ is at most 3. Thus Δ will have at most 3 linear factors: α,β, and γ . The possible Segre characteristic for a 3×3 matrix is [3], [2 1], [(2 1)], [1 1 1], [(1 1) 1], and [(1 1 1)]. Thus the invariant factor E_3 in each case is:

Case [3]: Then $\alpha = \beta = \gamma$ and thus $E_3 = \alpha^3$.
Case [2 1]: Then $\alpha = \beta$ and $E_3 = \alpha^2\gamma$.
Case [(2 1)]: Then $\alpha = \beta = \gamma$ and $E_3 = \alpha^2$.
Case [1 1 1]: Then $E_3 = \alpha\beta\gamma$.
Case [(1 1) 1]: Then $\alpha = \beta$ and $E_3 = \alpha\gamma$.
Case [(1 1 1)]: Then $\alpha = \beta = \gamma$ and $E_3 = \alpha$.

Therefore both λ-matrices will have the same E_3, therefore they are equivalent. □

Proposition 2 *If p is a monic factor of all i-rowed subdeterminants of a λ-matrix* $\mathbf{A}$*, it will be a factor of all i-rowed subdeterminants of every λ-matrix* $\mathbf{B}$ *which is equivalent to* $\mathbf{A}$

Proof $\mathbf{A} \sim \mathbf{B} \Rightarrow \mathbf{A} = \mathbf{PBQ}$ where $\mathbf{P}$ and $\mathbf{Q}$ are regular λ-matrices. The elementary transformation of interchanging two rows or columns have the effect on the i-rowed determinant of $\mathbf{A}$ of multiplying it by a nonzero constant, thus p is still a factor. The elementary transformation of multiplication of each element of a row (or a column) by the same nonzero scalar will have the effect on the i-rowed subdeterminants of $\mathbf{A}$ of multiplying it by a nonzero constant, thus p is still a factor. The elementary transformation of adding to the elements of some column j, the elements of some column k, each multiplied by a polynomial $\phi(\lambda)$. Any i-rowed subdeterminant which does not involve the j^{th} column will be unchanged. Any i-rowed subdeterminant which involves both the j and k column will remain unchanged. An i-rowed subdeterminant which involves the j^{th} column but not the k^{th} column may be written after the transformation in the form $a \pm \phi(\lambda)b$, where a and b are i-rowed subdeterminants of $\mathbf{A}$. Thus the proposition holds for any elementary transformation. So it holds for any product of elementary transformations. Thus it holds for any regular λ-matrices $\mathbf{P}$ and $\mathbf{Q}$. □

Proposition 3 *If* $\mathbf{A}$ *and* $\mathbf{B}$ *are equivalent* λ*-matrices of rank* r*, and* p_i *is the greatest common factor of the* i*-rowed subdeterminants* $(i \leq r)$ *of* $\mathbf{A}$*, then* p_i *is also the greatest common factor of the* i*-rowed subdeterminant of* $\mathbf{B}$*.*

Proof By proposition 2, p_i is a factor of all i-rowed subdeterminants of $\mathbf{B}$. If the i-rowed subdeterminants of $\mathbf{B}$ has a greater common factor than p_i, it would also be a factor of all the i-rowed determinants of $\mathbf{A}$; which is a contradiction. □

Proposition 4 *If* $\mathbf{A}$ *and* $\mathbf{B}$ *are equivalent* λ*-matrices then* $\mathbf{A}$ *and* $\mathbf{B}$ *will have the same standard form.*

Proof Since $\mathbf{A} \sim \mathbf{B}$, then every greatest common factor,$p(i)$, of the i-rowed determinants of $\mathbf{A}$ are the greatest common factor of the i-rowed determinants of $\mathbf{B}$. Thus the standard forms are the same. □

Proposition 5 *If* $\mathbf{A}$ *is a single variable* λ*-matrix then the standard form is the same as the canonical form.*

Proof The greatest common factor, $p(i)$, of the canonical form is just the product:

$$p(i) = \prod_{j=1}^{i} E_j$$

The d_i of the standard form is

$$d_i = \frac{p(i)}{p(i-1)} = E_i$$

Thus the standard form is the canonical from for single variable λ-matrices. □

Proposition 6 *If* $\mathbf{A}$ *and* $\mathbf{B}$ *are equivalent* λ*-matrices the* $\mathbf{A}$ *and* $\mathbf{B}$ *will have the same monic determinant and the same extended Segre characteristic.*

Proof $\mathbf{A} \sim \mathbf{B} \Rightarrow \mathbf{A} = \mathbf{PBQ}$ with the determinants of $\mathbf{P}$ and $\mathbf{Q}$ being constants. From $|\mathbf{A}| = |\mathbf{P}||\mathbf{Q}||\mathbf{B}|$, it follows that $\mathbf{A}$ and $\mathbf{B}$ have the same monic determinant. Since $\mathbf{A}$ and $\mathbf{B}$ have the same standard form, they will have the same extended Segre characteristic.

Lemma 2 *If two* $\boldsymbol{\lambda}$*-matrices,* $\mathbf{A}$ *and* $\mathbf{B}$*, are equivalent and are associated with families of coplanar conics then there exists two regular* $\boldsymbol{\lambda}$*-matrices,* $\mathbf{M}$ *and* $\mathbf{N}$*, of degree zero such that* $\mathbf{B} = \mathbf{MAN}$.

Proof There exists two regular $\boldsymbol{\lambda}$-matrices, $\mathbf{P}$ and $\mathbf{Q}$, such that $\mathbf{B} = \mathbf{PAQ}$. Let $\mathbf{P} = \tilde{\mathbf{P}} + \mathbf{P_0}$ and $\mathbf{Q} = \tilde{\mathbf{Q}} + \mathbf{Q_0}$ where $\mathbf{P_0}$ and $\mathbf{Q_0}$ are degree zero (scalar matrices) and $\tilde{\mathbf{P}}$, $\tilde{\mathbf{Q}}$ are matrices with each entry either zero or a polynomial of degree greater of equal to one. Then

$$\begin{aligned} \mathbf{B} &= \mathbf{PAQ} = (\tilde{\mathbf{P}} + \mathbf{P_0})\mathbf{A}(\tilde{\mathbf{Q}} + \mathbf{Q_0}) \\ &= \tilde{\mathbf{P}}\mathbf{A}(\tilde{\mathbf{Q}} + \mathbf{Q_0}) + \mathbf{P_0}\mathbf{A}\tilde{\mathbf{Q}} + \mathbf{P_0}\mathbf{A}\mathbf{Q_0} \end{aligned}$$

Since $\mathbf{A}$ and $\mathbf{B}$ are associated with families of coplanar conics, then all entries in the two matrices must either be zero or of degree one. Furthermore since all entries in $\tilde{\mathbf{P}}$ and $\tilde{\mathbf{Q}}$ are either zero or of degree at least one, the product of $\tilde{\mathbf{P}}$ or $\tilde{\mathbf{Q}}$ with $\mathbf{A}$ must be zero or of degree greater than one. Since $\mathbf{B}$ is degree one, the terms $\tilde{\mathbf{P}}\mathbf{A}(\tilde{\mathbf{Q}}+\mathbf{Q_0})+\mathbf{P_0}\mathbf{A}\tilde{\mathbf{Q}}$ must reduce to zero. Therefore $\mathbf{B} = \mathbf{P_0}\mathbf{A}\mathbf{Q_0}$. Let $\mathbf{M}$ and $\mathbf{N}$ be $\mathbf{P_0}$ and $\mathbf{Q_0}$, respectively.

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Table 7. Invariants of Four conic families in Figure 3.

Invariant	Figure 3(a)	Figure 3(b)	Figure 3(c)
$\mathcal{I}_1^4$	-0.0785	-0.0559	-0.0813
$\mathcal{I}_2^4$	-0.366	-0.144	-0.288
$\mathcal{I}_3^4$	-0.586	-0.347	-0.393
$\mathcal{I}_4^4$	-0.908	-0.958	-0.866
$\mathcal{I}_5^4$	-0.439	-0.261	-0.459
$\mathcal{I}_6^4$	0.184	0.0927	0.168
$\mathcal{I}_7^4$	-0.0959	-0.0635	-0.0675
$\mathcal{I}_8^4$	-0.875	-1.02	-0.803
$\mathcal{I}_9^4$	0.0130	0.00297	0.00411
$\mathcal{I}_{10}^4$	0.750	0.243	0.602
$\mathcal{I}_{11}^4$	0.233	0.116	0.198
$\mathcal{I}_{12}^4$	-0.872	-0.977	-0.748

Table 8. Invariants $\mathcal{I}_1^6$ and $\mathcal{I}_1^7$ for figures 4, 5, and 6(a).

Figure	$\mathcal{I}_1^6$	$\mathcal{I}_1^7$
Figure 4(a)	0.297	-
Figure 4(b)	0.279	-
Figure 5(a)	0.865	0.000275
Figure 5(b)	1.10	0.000455
Figure 6(a) - Crane	0.283	-
Figure 6(a) - Ammo Carrier	2.15	0.0000827

Table 9. Distance Measure between values of invariant of Table 8

	Figure 6(a) Crane	Figure 6(a) Ammo Carrier
4(a)	0.0008	0.0228
4(b)	**0.0006**	0.0242
5(a)	0.0144	0.0092
5(b)	0.0170	**0.0066**
6(a) - Crane	-	0.0236
6(a) - Ammo Carrier	0.0236	-

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